WHAT CAN BE LEARNT BY STUDYING THE ADULT MODAL AGE AT DEATH?

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ABSTRACT

Taking all the data available in the Human Mortality Database (HMD), this study illustrates the interest of a mode-oriented approach to longevity research, focusing on the most common longevity. Several indicators built from the adult modal age at death (M) are introduced to describe the adult longevity and its changes over time, examining whether an increase in the most frequent age at death is always accompanied by a compression of the mortality occurring above it. Comparative analysis with standard mortality measures (i.e. life expectancy at birth (e(0)), median, life expectancy at age 65 and the logarithm of the geometric mean of age-specific death rates between ages 65 and 99) is also presented. In this study, we show that M is useful not only as a measure of longevity but also in understanding and applying major mathematical models of mortality trajectory (i.e. the Gompertz, logistic, Weibull, Lexis models), using M as a parameter and investigating characteristics of M-related measures in those models.

A. INTRODUCTION

We begin this study of adult longevity with a literature review (section B) concerning the distribution of deaths by age (i.e. the death curve or d_x in a life table) since Lexis (1878) and Pearson (1897) through Greenwood and Irwin who challenged the Lexis's proposal of stability of the normal lifetime in 1939. Later, in the 1950, Clarke suggested again that the natural life spans should form an invariant distribution which represents the ultimate or limiting form of the curve of deaths. His work has been further developed by Benjamin who challenged his proposal of invariant distribution of senescent deaths as Greenwood and Irwin did for the Lexis's proposal (Benjamin 1959, 1963). At the beginning of the 21st century, Kannisto (2001) revived the interest for the d_x approach and developed his hypothesis of an "*invisible wall*" to the extension of human longevity. Following his study, Cheung (2003) and her colleagues (Cheung et al. 2005a and 2005b) applied the Lexis-Kannisto model to several national time-series and developed associated indicators. Based on these previous studies, we believe that, when looking at changes in human longevity (past and future), we have to break down the longevity question into two parts; the first part or the first question being: 'How many newborn are becoming adults?' and the second part being 'How long are adult life durations?'

In section C, we present empirical observations of the modal age at death and its associated indicators using all the data available in the Human Mortality Database (HMD) by February 2006; i.e. 5032 period life tables for 26 countries, East and Western Germany East taken together, starting with Sweden in 1751.

In section D, we perform some comparative analysis with standard mortality measures (i.e. two measures of overall mortality, life expectancy at birth (e(0)) and median age at death, and two measures of old-age mortality, life expectancy at age 65 and the logarithm of the geometric mean of age-specific death rates between ages 65 and 99) .

In section E, we develop some mathematical formula to re-express popular mortality models (i.e. Gompertz, logistic, Weibull, and Lexis models) using M as a parameter, and investigate characteristics of M-related measures in those models. We claim that our approach allows a better understanding of the conventional mortality models

B. LITERATURE REVIEW

Since a long time, actuaries and researchers in bio-medical science have been concerned with the length of human life. Several life table functions, such as the life expectancy at birth (e_0), have been used to examine this question. More specifically, the force of mortality (μ_x) or the probability of dying (q_x) have been used to analyse the mortality trajectory deceleration observed at higher ages. The proportion of survivors (l_x) has been used to assess whether the survival curve is becoming more rectangular and whether the deaths are compressed into a narrower band of ages. The final objective of these studies was to discover whether a fixed limit to human longevity exists.

Looking for a law of mortality, by fitting mathematical functions to the death rates, has been the main approach in this quest since de Moivre (1756) and Gompertz (1825). For instance, Thiele (1871) had proposed complex mathematical relationships between the ages at death and the force of mortality (i.e. a decreasing Gompertz curve to represent childhood mortality, a normal curve in the middle range of ages, and another Gompertz curve to represent old-age mortality).

However, parallel attention has been given to the distribution of deaths by age (i.e. the death curve or d_x in a life table) as an alternative operand. Thus, Lexis pioneered in 1878 the concept of normal life duration, characteristic of a natural and ageing lifetime. He divided the distribution of deaths by age in three parts: (1) a J-curve right after birth corresponding to infant deaths; (2) the normal deaths around the late modal age at death which obey the law of accidental errors and reflect a natural lifetime; and (3) a transitional region where adult premature deaths partly overlap with the normal deaths (See figure 1, Lexis's diagram). The modal age at death of the second distribution (i.e. the late modal age at death) represents the most central and natural characteristic of the human longevity. All deaths occurring at and above this mode are regarded as "normal" and account for the right-hand side of a normal distribution. The hypothetical left-hand side of distribution, below the mode, is disentangled from premature adult deaths by using the symmetrical property of the normal distribution (Lexis 1878). This proposal of Lexis supposed that all or almost all deaths occurring at the mode and above can be considered as natural or ageing-related by contrast to the premature deaths. It was warmly praised by statisticians and economists near the end of the nineteenth century (Bertillon 1878; Bodio 1887; Elderton 1903; Levasseur

1891; Pareto 1896; Perozzo 1879) (see Véron and Rohrbasser 2003 for more details on Lexis's approach) and by biologists who later opposed the Gompertzian to the Lexian model (Greenwood and Irwin 1939).

FIGURE 1 ABOUT HERE

Pearson also drew careful attention to the distribution of deaths by age in his book *The Chances of Death,* published in 1897. He described this distribution as a chance distribution following perfectly clear mathematical laws defined by a certain skewness and precision. He considered that the distribution of deaths by age is not just one simple frequency distribution, but is made up of five components (i.e. old age, middle life, youth, childhood, and infancy mortality (see figure 2, Pearson's mortality curve)) with different degrees of skewness and precision for each component. He highlighted that the frequency of death at later ages must depend on the incidence of death at earlier ages. He also noticed that whatever be the degree of skewness, practically the whole of the distribution falls within a range of three times the standard deviation taken on either side of the mean. A common feature put forward by Lexis and Pearson is the use of the distribution of deaths by age and its modal values to distinguish different components of mortality, corresponding to different age ranges.

FIGURE 2 ABOUT HERE

However, although the late modal age at death has been used earlier to characterize the natural and normal life span, life expectancy at birth was considered as the best index of the life span during the 20th century (Dublin 1923). It is still, at present, the most popular and widely used index providing a robust summary measure of population's health. Indeed, during the early stages of the epidemiological transition when infant and child mortality were dominant, using life expectancy at birth as longevity indicator was advantageous as this indicator is highly sensitive to infant mortality and premature deaths. Today in the low mortality countries, while the majority of deaths occur at older ages, focusing on life expectancy can omit some important information. The advantages of the late modal age at death to study the adult longevity were regularly underlined during the 20th century (Elderton 1903; Greenwood and Irwin 1939; Gumbel 1937).

Developing the concept of normal life duration, Lexis used empirical data for the early 1880s to estimate the normal lifetime (i.e. the modal age at death). It was between 70 and 72.5 years in Central European countries, 75 years in Sweden and 78 years in Norway (Lexis 1903). Lexis claimed that this age should be stable over time. Gumbel provided similar values in 1937, ranging from 70.6 years for males and 73.7 years for females in Switzerland in 1901-1910 to 78.5 years for males and 78.8 years for females in the United States (white population) at the same period (Gumbel 1937). However, referring to the work of Freudenberg (1934) for the German Life Tables, covering the period from 1871-1881 to 1924-1926, and to Elderton and Oakley for four English Life Tables, Greenwood and Irwin argued that the late modal age at death was increasing over time, challenging the Lexis's proposal for stability of the normal lifetime. They underlined that the modal lifetime was a variable quantity and that minimizing the unfavorable environmental factors would lead some people, who otherwise would have met a premature death, to reach the modal age at death (Greenwood and Irwin 1939).

Other scholars such as Phillips (1935, 1954) and Beard (1950) paid attention to the distribution of deaths by age. Clarke (1950), for instance, distinguished between "anticipated" and "senescent" deaths. Like Lexis, he considered that the ages at death in the latter group were a measure of natural life spans. He argued that the observed decline in mortality cannot be associated with an extension of the life span as there is no reliable indication that any mortality improvement has occurred at advanced ages, say over the age of 85 years. Lives have been saved only at younger age, allowing more people than formerly to live out their full span or a longer portion of it. In this condition, Clarke considered that all deaths at age 80 and over are senescent deaths corresponding to the natural termination of the life spans. On the other hand, anticipated deaths correspond to all deaths, whether from accident, disease or any other cause, which are anticipations of the natural termination of life. According to Clarke, every individual carries with him his natural term of life, i.e. the age beyond which it is impossible for him to survive. Thus the distribution of "senescent" deaths by age should form a frequency distribution, in the same way as height and head-breath or all other hereditary characteristics. It should represent the ultimate or limiting form of the curve of deaths, suggesting again that the natural life spans are invariant (Clarke 1950).

Although similarity can be found in many aspects with the proposal of Lexis, it is noteworthy that the proposal of Clarke is theoretically independent of the modal age at death. The value of 80 years, proposed in his paper published in 1950, is 6 years higher than the observed modal age at death for males in the last available English life table at that time (ELT 10, 1930-1932). Moreover, Clarke acknowledged that the choice of the age of 80 years may be too young, especially looking at the mortality level experienced in the last life tables. From a methodological point of view, Clarke claimed that if the aim of the research is the natural life span, d_x should be the primary function to look at, while q_x or mu_x should be focused if the aim of the study is the anticipated mortality (Clarke 1950).

The work of Clarke has been further developed by Barnett (1955, 1958) and Benjamin (1959, 1963, 1964, 1982 and 1988). In particular, in 1959, Benjamin proposed for simplicity to assume that the "senescent" deaths are symmetrically distributed around the late modal age at death. Considering that all the deaths occurring at the mode and above are senescent deaths, this proposal allows him to disentangle the "senescent" deaths from the "anticipated" deaths, exactly like Lexis did several decades before him. The left-hand side of the distribution of "senescent" deaths by age mirrors exactly the right-hand side. Then considering that the late modal age at death in the *English Life Tables No. 1, 8* and *11*, as well as the proportion of deaths falling in the area of "senescent" deaths, have increased over time, Benjamin argued that Clarke's invariant distribution of senescent deaths hypothesis is erroneous (Benjamin 1959, 1963 and 1964; Clarke 1963).

FIGURE 3 ABOUT HERE

Kannisto revived, at the beginning of the 21st century, the interest for the d_x approach. Using the Lexis's concept of normal life durations, he developed his hypothesis of an "*invisible wall*" to the extension of human longevity on the basis of a negative relationship between the modal age at death (M) and the standard deviation of the ages at death occurring above it (SD(M+)). He found a negative correlation between M and SD(M+) in cross-sectional data from fifteen low mortality countries for the period 1990-1995 and in time-series data for the United States. According to Kannisto, M and SD(M+) give a good account of the adult longevity under a given mortality regime. While M is increasing, it is not simply sliding to the right. Instead, its right-hand slope is being flattened vertically as if it was meeting an invisible wall. He contended that the rising trajectory of mortality in the highest age groups forms a barrier but only in a relative sense, offering stiffer resistance to further progress without setting any definite limit to life span (Kannisto 2001). In his study, he found a fairly universal pattern in the distribution of deaths at old age which is consistent with the observation of Lexis that the distribution of deaths by age above the modal age at death approximates the second half of a normal distribution (Kannisto 2001).

Building on this approach, two associated indicators were proposed in the form of M-/+*k*SD(M+) to indicate the shortest and longest normal life durations (Cheung 2003; Cheung et al. 2005a, 2005b). Preliminary results confirmed the observations of Kannisto, indicating that a compression of mortality occurred over time. According to hypothesis of Kannisto, the extension of human longevity might be meeting an increasing resistance. (Kannisto 2001). Empirical time series of the maximum reported age at death (MRAD), from 1876 to 2002 in Switzerland, suggest a value of 3.2 for *k* (Cheung et al. 2006), and, from 1950 to 1999 in Japan, values of 3.5 for males and 3.6 for females (Cheung and Robine 2006).

Although the utility of the modal age at death is now well acknowledged in studies of senescence and longevity because it is determined by adult mortality only (Horiuchi 2003), its right determination is crucial. Indeed, it can be often located at different ages within the distribution of deaths by age due to certain flatness of the curve in the modal region (Kannisto 2001). Already in 1902, Pearson underlined that fallacious modal values could be chosen by a casual inspection. He suggested that the satisfactory way of determining the mode was to interpolate a curve through the tops of ordinates (Pearson 1902).

C. EMPIRCIAL OBSERVATIONS

In the first part of this section, we used all the data available in the Human Mortality Database (HMD) by February 2006 ; i.e. 5032 period life tables for 26 countries (East and Western Germany East taken together), starting with Sweden in 1751.

To examine the question of how many newborn are becoming adults, we took the number of survivors at age 18 (i.e., l(18)) as a first estimation of the proportion of newborn becoming adults. Adulthood is more a social concept than a biological one and "18 years" is the legal age to vote in many low mortality countries but other ages from 15 to 25 are possible. We guess that they will provide about the same information than 18. From a biological point of few, we can choose other ages such as the age at which the mortality rate is the lowest, indicating the strongest resistance to death or the age at puberty, indicating sexual maturity but the former is close to the age of 11 years and badly linked to adulthood and the later is difficult to assess.

FIGURE 4 ABOUT HERE

Figure 4 illustrates one of the most important change occurred during the demographic transition. Before the transition less that 70% of the newborn become adults. Then, from about 1820 in Sweden to 1920 in Spain, the proportion of newborn becoming adults rises above 70% to reach 99.5% today in the main low mortality countries, with little changes since 1990. Therefore, from now onwards, the longevity question becomes merely how long are adult life durations?

How long are adult life durations?

As in previous work, in this study we describe the adult life durations with the help of three indicators: M, the most frequent age at death; SD(M+), the standard deviation of the ages at death above M; and M+*k*SD(M+) as indicator of the highest life durations.

In this explanatory study we directly use the observed M in the period life tables without any fitting (correction), we estimate SD(M+) with the simple formula SD(M+) = e(M)*1.25 as suggested by Kannisto (2001) and we compute M+*k*SD(M+) for *k*=3, *k*=3.5 and *k*=4.

FIGURE 5 ABOUT HERE

Figure 5 shows that the most frequent adult life durations fluctuated between 70 and 75 years before the demographic transition. During the first part of the demographic transition, from 1840 to 1940, the picture is quite blurred with values fluctuating between 70 years (Spain) and 80 years (Norway and Sweden). After 1940, a steady increase in the modal age at death is observed in all low mortality countries (Australia, Western and Nordic Europe, North America and Japan). The noise created by the Eastern European countries does not confuse the general trend.

FIGURE 6 ABOUT HERE

Figure 6 shows that the standard deviation of the ages at death above M (SD(M+)) fluctuated between 8 and 10 years without any trend before the demographic transition. Again, during the first part of the demographic transition the picture is blurred with an increase in the fluctuations. However, after 1940, a clear decreasing trend emerged in the main low mortality countries. In the last period, SD(M+) reached 6 years or less in several countries (such as Switzerland).

Note that Iceland presents extremely low and fluctuating values for M, complemented by extremely high and fluctuating values for SD(M+), probably due to its population size.

FIGURE 7 ABOUT HERE

Figure 7 suggests, with M+3SD(M+) as indicator of the highest life durations, that the highest life durations decreased from 1751 to 1801 in Sweden. In the first part of the demographic transition the picture is blurred with an indicator fluctuating between 95 and 105 years but since the 1940s or the 1950s the increasing trend is obvious in almost all countries with all the values being postponed by about 5 years in 2001 and from now on reaching 110 years. Iceland displays again extremely high fluctuations in this indicator.

FIGURE 8 ABOUT HERE

Figure 8 shows the same trend, with M+4SD(M+) as indicator of the highest life durations, than figure 7. However, the trend is being flattened. Globally, the highest life durations fluctuated around 110 years from 1751 to 2001, suggesting that with k=4, the product of M+kSD(M+) is offsetting the increase in M.[1] On the other hand,

[1] Note that if SD(M+) and M are linearly related such that SD(M+)=C-M/4, then

the estimated values are much higher than the observed values before 1950, suggesting that the empirical values of *k* are below 4, probably between 3 and 3.5 as suggested by previous study (*k*= 3.2 in Switzerland and k = 3.5 in Japan) (Cheung et al. 2006; Cheung and Robine 2006).

Some empirical relationships

Beyond significant fluctuations, figure 9 suggests that the probability of dying at the age, corresponding to the most frequent age at death (i.e., q(M)), increases at least since 1950. This last observation is concomitant with the decrease in SD(M+) during the same period.

FIGURE 9 ABOUT HERE

Figure 10 is clearly related to figure 4. Before 1851, before the demographic transition a small proportion of people reach the age corresponding to the modal life durations, about 20%. During the first part of the transition, between 1851 and 1951, this proportion rose to 40%. This is mainly due to the fact that more newborn reach adulthood. However since 1950, this proportion stops increasing. According to previous works by John Pollard and our own work, if adult mortality follows a Gompertzian trajectory – and in an absence of infant mortality, lx at M will level off at about 36.8% corresponding to exp(-1) (see figure 24, changes in the age at l(x)=e(-1)). If adult mortality follows a logistic trajectory, and in an absence of infant mortality, lx at M will level off between 38% and 41%, according to a sensible range of values for the logistic parameters (see section E).

FIGURE 10 ABOUT HERE

Beyond the general trend for l(M), reaching a plateau after WWII, individual national values scattered from 30% to 50% in 2001 as well as in 1951.

FIGURE 11 ABOUT HERE

Figure 11 shows three periods in the increase in the number of deaths occurring in the nominal year corresponding to the modal age at death (d(M)). During the first period, before the demographic transition, about 2% of deaths occurred in this single year of age. During the first part of the demographic transition, from 1851 to 1951, d(M) doubled from 2% to 4%. Since 1951, d(M) goes on increasing but at a much slower

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M+4SD(M+) is constant at 4C. As shown in figures 12 and 21, the relation between SD(M+) and M is curvilinear, but can be approximated by a straight line fairly well.

pace. The number of deaths, d(M) is the product of l(M) by q(M). Since we have seen in figures 9 and 10 that l(M) is currently in the region of 0.4 and that q(M) is currently in the region of 0.1, we can expect to find d(M) to be in the region of 0.04 or slightly more. This is confirmed in figure 11. However, figure 11 also shows the historical trend which was much influenced by the levels of infant and child mortality, affecting the values of l(M).

Eventually, the last figure of this first part, figure 12 illustrates the correlation between M and SD(M+), suggesting a curvilinear relationship.

FIGURE 12 ABOUT HERE

In the second part of this section, we present the same figures but after having eliminated several countries. Obviously, there is a lot of noise in the HMD and, after having presented the global pattern, we would like to focus on the low mortality countries and the most probable patterns.

We started with 10 countries, Denmark, England and Wales, France, Japan, the Netherlands, Norway, Sweden, Switzerland, USA and Iceland (see figure 13).

FIGURE 13 ABOUT HERE

Due to a small population size, Iceland shows great fluctuations. Thus, we eliminated this country in the following analysis.

Even if Norway presents quite high values in the second part of the 19th century and, on the opposite, Switzerland quite low values, compared with figure 5, most of the noise has been removed on the figures 14 and 15, showing the general pattern of the changes of the modal age at death and the standard deviation of the ages at death above M.

FIGURE 14 ABOUT HERE

FIGURE 15 ABOUT HERE

Then the observation of changes in M+3SD(M+), presented at the figure 16, led us to eliminate the US which is displaying an unexpected trend for this indicator of the highest life durations. The combination of M and SD(M+) highlights a peculiar trend in the US, partially visible on the previous figures 5 (changes in M) and 6 (changes in SD(M+)). This raises again questions about the quality of the US data (Kannisto 1994).

FIGURE 16 ABOUT HERE

FIGURE 17 ABOUT HERE

Figures 17 and 18 suggest that, excepting Japan, *k* is probably lower than 3.5. To further specify k, we calculated the average maximum reported age at death for England and Wales from 1911 to 1999 and Sweden from 1861 to 2003 (See figure 25). The results show that England and Wales has similar values for both sexes than Japan (k=3.5) and Sweden has same values for both sexes than Switzerland (k=3.2). One reason for these differences in the empirical value of k lies in the fact that Japan and England and Wales are much larger countries than Sweden and Switzerland. Population size matters in the determination of k. For example, a theoretical calculation by a method suggested by Thatcher (1999) shows that in a large country where 50 people reach the age of 105 years we can expect that one or two will reach the age of 110 years, but in a small country where only 5 people reach the age of 105 years it is unlikely that any of them will reach the age of 110 years. If we assume that deaths above the mode follow the Lexis model, a theoretical calculation shows that if a small country is of a size which produces a value k = 3.2, then it is perfectly possible that a country which has a population five times as large (but is otherwise demographically similar) can produce a value k = 3.6. Thus differences in k are to be expected between countries of different sizes.

FIGURE 18 ABOUT HERE

FIGURE 19 ABOUT HERE

Beyond remaining fluctuations, figure 19 suggests a steady increase in q(M) during the 20th century from about 9%, in average, in 1900 to about 12% in 2000. In the Gompertz model there is a very simple relationship, which shows that the force of mortality at the mode, denoted by $\mu(M)$, is equal to the Gompertzian ageing rate, as represented by the parameter b (see section E1 below, equation (4)). A typical modern value would be b = 0.12. For this value the relationship shows that $\mu(M) = 0.12$, which implies that q(M) = 0.11, which is reasonably in line with the data plotted in figures 9 and 19. This is a reassuring agreement between theory and observation. However, the figures give much new information about the past values of q(M), which in turn implies information about the past trend in the Gompertzian ageing rate.

FIGURE 20 ABOUT HERE

Figure 20 shows significantly less fluctuations in the number of survivors at M than in figure 10. However, l(M) remains quite scattered around 40% in the last fifty years.

FIGURE 21 ABOUT HERE

Eventually figure 21 illustrates the relationship between M and SD(M+) in the 8 selected countries: Denmark, England and Wales, France, Japan, the Netherlands, Norway, Sweden and Switzerland.

The value of SD(M+) depends on the distribution of age at death above the mode, which we shall show in section E are well fitted in life tables by both the logistic and the Lexis models. The Lexis model implies a particularly simple relationship between SD(M+) and the force of mortality at the mode, which is denoted by $\mu(M)$[2] (see figure 26). This simple relationship can be written as $\mu(M) = 0.798 / SD(M+)$ (Proof is given in section E). We know from results given above that modern values of $\mu(M)$ are in the region of 0.12, so the simple relationship leads us to expect that SD(M+) will be in the region of 6.66 years. This appears to be well in line with figures 6 and 15, so that there is again a reassuring agreement between model and observation.

However, figures 12 and 21 go much further. They confirm very clearly and with compelling evidence the existence of the trend which was first noted by Kannisto (2001). SD(M+) has been falling while M has been rising. The ages of death above the mode have become more compressed. What was the reason?

The simple equation above shows that there is an inverse relationship between SD(M+) and $\mu(M)$ as shown in figure 26. As one rises, the other will fall. However, this does not mean that one causes the other. In fact, these are both statistics which have been calculated from the ages at death. Moreover, in a life table the ages at death are entirely determined by the death rates. Ultimately, it is changes in the death rates which cause both the rise in M and the rise in $\mu(M)$ and the fall in SD(M+).

[2] We have used the approximation $\mu(M) = -\ln(1-q(M))$

The basic fact is that death rates in the highest age groups have not been falling as fast as the death rates in the age groups below them. This means that although age-specific death rates have fallen at all ages, they have fallen more at low ages than at high ages, so that the slope of a line fitted to them has become steeper. In the terminology of section E, the ageing rate has risen. In the Gompertz model, this means that μ(M) will rise. In the Lexis model, this means that SD(M+) must have fallen. Perhaps, though, it does not really need models to show that if the ageing rate increases, then the ages at death will be compressed.

The next question is in practice why death rates in the highest age groups did not fall faster during the periods when compression occurred. It is no doubt a matter for controversy. And perhaps sooner or later that it is absolutely possible for them to fall faster. This can be explained by the fact that new drugs (such as the discovery of sulphonamide and penicillin and the other antibiotic drugs, and the discovery of the cure for tuberculosis) and the improvements in surgery and medical technology do have reduced the general level of mortality at young and middle ages, and so have driven up the mode. However, at very high ages, a considerable proportion of people have conditions that do not respond to drugs and surgery, at least to the same extent as younger people. The body deteriorates. Measurements of vital capacity, maximum heart rate, maximum oxygen consumption and the rate of cell renewal all decline with age. Recovery times are longer. There are also irreversible changes, like the loss of brain cells and lung tissue. In the phrase of Gompertz, a man's power to avoid death is gradually exhausted. Thus it is not surprising that drugs and surgery can only cure a smaller proportion of people at very high ages than at younger ages.

D. COMPARATIVE ANALYSIS WITH OTHER MORTALITY MEASURES

The most widely used measure of central tendency of death distribution in period life tables is the life expectancy at birth (e(0)). It can be strongly affected by high mortality at young ages. So is the median age at death, another major mortality measure of central tendency. On the other hand, usually the late modal age at death (M) is determined solely by mortality at old ages. Substantial changes in infant and child mortality will not alter M at all.

Figure 22 displays trends in M (estimated from smoothed d(x) curves), median and e(0) by sex for France and Japan. It reveals that levels and trends of these three central tendency measures are notably different. Numerically, they are ranked, in the descending order, as M, median and e(0), because the relatively large number of deaths among infants and at very young ages as well as the left-skewed pattern of the bell-shaped distribution of adult deaths make e(0), and to a lesser extent, the median, smaller than the mode.

FIGURE 22 ABOUT HERE

Differences among those measures were considerable when mortality at young ages was high, but their differences remain notable even in recent years: M is higher than the median by a few years, and the median is higher than e(0) by a few years. e(0) for Japanese females is known to have surpassed 85, but their M is actually over 90. Although e(0) is usually considered as "the typical age at death," particularly among the general public, the true typical age is significantly higher than e(0) if the most frequent age at death should be considered as typical (Kannisto 1996).

Trends of the three indicators are noticeably different, too. In France, both e(0) and the median age increased substantially during the first half of the twentieth century, but M remained nearly constant for males and increased only slightly for females. A remarkable increase in M started during the second half of the twentieth century (apparently around 1950 for females and around 1970 for males), whereas the increases of e(0) and the median age slowed down a little, and the three measures has been ascending at comparable pace in the last few decades. In brief, during the twentieth century, the increases of e(0) and median decelerated, but in contrast, the increase of M accelerated, and the turning points appear to have been in the third quarter of the century.

For Japan, highly reliable mortality estimates are limited to the second half of the century, in which the trends were fairly consistent with those of France. In the 1960s and 1970s, the rises of e(0) and median decelerated but M continued to ascend linearly. During the last few decades, the three measures exhibit parallel linear increases.

Thus, M, median and e(0) tell us different stories about the recent history of lifespan extension. The main reason for the differences seems to be the fact that, unlike the other two, M is essentially an overall measure of old-age mortality. Figure 23 compares M with two other indicators of old-age mortality, the life expectancy at an old age (65 in this analysis) and the logarithm of geometric mean of age-specific death rates at old ages (65-99 was chosen). Data from France and Japan indicate that M is highly linearly correlated with both of the two measures. Therefore, these three indicators are expected to show very similar trends of old-age mortality.

FIGURE 23 ABOUT HERE

However, in practice, M is particularly useful because it is obtained without selecting a specific age range. In figure 23, e(65) was adopted as an old-age mortality measure. This means that 65 was chosen as the starting point of old age. In order to calculate the geometric mean of death rates, an age range has to be specified. In both cases, the decision is somewhat arbitrary, and an appropriate range of "old age" may shift upward as old persons in later generations become healthier and less frail. M is free of this problem.

Thus the discussion and data analysis in this section seem to suggest that M is a useful measure of longevity, particularly for economically and technologically developed countries in which mortality improvements are mainly due to declines in old-age mortality.

E. THE MODAL AGE AT DEATH IN MORTALITY MODELS

The later modal age at death (M) is useful not only as a measure of longevity but also in understanding and applying major mathematical models of mortality trajectory. In what follows, we will re-express popular mortality models using M as a parameter, and investigate characteristics of M-related measures in those models.

1. Re-expression of mortality models using the modal age at death

Most widely used models can be formulated using M. To our knowledge, this special advantage is not found for the life expectancy or the median age at death.

The force of mortality at age *x* in the Gompertz, logistic and Weibull models are conventionally expressed as follows:

Gompertz: $\mu(x) = ae^{bx}$ (1.)

Logistic (with three parameters): $\mu(x) = \dfrac{ae^{bx}}{1+(a/g)e^{bx}}$, (2.)

Weibull: $\mu(x) = ax^{b}$ (3.)

These models can also be expressed using M:

Gompertz: $\mu(x) = be^{b(x-M)}$ (4.)

Logistic (with three parameters): $\mu(x) = \dfrac{be^{b(x-M)}}{1+(b/g)e^{b(x-M)}}$ (5.)

Weibull: $\mu(x) = \dfrac{b}{M}\left(\dfrac{x}{M}\right)^{b}$ (6.)

Note that although the same symbols (*a* and *b*) indicate different things in different models, each has comparable meanings across those models. Conceptually, in each model, *b* is the parameter representing some "aging rate," i.e., the extent to which mortality rises with advancing age, though mathematically it differs among the models: the rate of exponential, logistic, or polynomial increase. *a* is exactly or approximately the force of mortality at a "reference age" such as 0 and 1: $\mu(0)=a$ in the Gompertz model, $\mu(0)=a/\{1+(a/g)\}\approx a$ in the logistic model, and $\mu(1)=a$ in the Weibull model. *g* in equations (2) and (5) is the upper bound of logistic growth. *a* ,*b* and *g* are assumed positive.

For each model, the Makeham version could be set up as shown below, by assuming that adult mortality is the sum of premature mortality, which is assumed constant over age and denoted by *c* below, and senescent mortality, which is represented by the original form:

Gompertz-Makeham: $\mu(x) = c + be^{b(x-M_s)}$ (7.)

Logistic-Makeham: $\mu(x) = c + \frac{be^{b(x-M_s)}}{1+(b/g)e^{b(x-M_s)}}$ (8.)

Weibull-Makeham: $\mu(x) = c + \frac{b}{M_s}\left(\frac{x}{M_s}\right)^b$ (9.)

These Makeham versions use the modal age at death from senescent mortality (M_s), but usually, when a Makeham version is fitted to mortality data, premature mortality is almost negligible compared to senescent mortality around M. Therefore, M_s is nearly equal to (though slightly higher than) M.[3]

In each of the models, *a* in the conventional forms (equations (1), (2) and (3)) is replaced by M in the new forms (equations (4), (5) and (6)). In each case, M is intuitively clearer than *a*. For example, let us compare equations (1) and (4) for the Gompertz model as an example. M seems more easily interpretable than *a* in the following three aspects. Firstly, the Gompertz model is fitted to death rates at adult ages, and *a* is the force of mortality at age zero extrapolated from adult ages. Thus the interpretation of *a* is not as straightforward as that of M, because the observed neonatal death rate is considerably higher than *a*. Secondly, the value of *a* is very small and most researchers do not have a clear idea about its plausible range, but they have a clear idea about the plausible range of the most frequent age of adult deaths. Finally, when mortality schedules of two populations are compared, a paradoxical result about *a* could be obtained. Although *a* is generally considered to be a parameter

[3] For example, according to the logistic-Makeham model fitted to mortality data for Swedish women aged 55-95 in 1973-1977 (Horiuchi and Coale 1990), M=84.3 and M_s=84.6. This proximity is not surprising, because at very old ages, senescent mortality is usually high enough to make premature mortality almost negligible. It is estimated from the fitted model that 98 percent of $\mu(M)$ is due to senescent mortality, and only 2 percent is due to premature mortality.

indicating the overall level of mortality, it is possible for a population with higher adult death rates to have a lower value of a than the other population, if b is substantially different between the two schedules. In contrast, a lower value of M almost always indicates higher mortality rates at old ages at which many deaths occur.

In what follows, we will derive the M-related expressions (equations (4), (5) and (6)) from the corresponding conventional expressions (equations (1), (2) and (3), respectively). All of the derivations are based on a fundamental relation shown by Pollard (1991): at the modal age, the force of mortality and the life table aging rate (Horiuchi and Wilmoth 1997) are identical, i.e.,

$$\mu(M) = k(M) \qquad (10.)$$

where $k(x) = d \ln \mu(x)/dx$. This is obtained by differentiating $d(x) = l(x)\mu(x)$ with respect to x, setting x=M, and making use of the fact that the derivative of $d(x)$ at M is zero. The life table aging rate (LAR) for each model is as follows (Horiuchi and Coale 1991):

Gompertz: $k(x) = b$ (11.)

Logistic: $k(x) = \dfrac{b}{1+(a/g)\,e^{bx}}$ (12.)

Weibull: $k(x) = b/x$ (13.)

For the Gompertz model, by setting x=M in equations (1) and (11) and substituting them into equation (10), we get $a = be^{-bM}$. Substitution of this into equation (1) leads to equation (4). A similar derivation works for the logistic model. By setting x=M in equations (2) and (12) and substituting them into equation (10), we obtain $a = be^{-bM}$ again, which is substituted into equation (2), resulting in equation (5). For the Weibull model, by setting x=M in equations (3) and (13) and substituting them into (10), we have $a = b\,M^{-(b+1)}$. Substitution of this into equation (3) leads to equation (6).

2. M-related measures in mathematical models

In this section, mathematical expressions of M, $\mu(M)$, $l(M)$ and $d(M)$ for the Gompertz, logistic, and Weibull models are shown in terms of their conventional parameters. The expressions for $l(M)$ and $d(M)$ should be taken with caution, because they are obtained assuming that the age trajectory of mortality throughout the entire lifespan follows the model, which actually fits well mortality at adult ages only. Thus, if the model fits mortality above age 30, the analytical expressions for *l(M*)* and *d(M*)*, where *M*=M-30*, should be close to observed values of *l(M)/l(30)* and *d(M)/l(30)*.

Gompertz model

Pollard and his colleagues have investigated characteristics of *M* in Gompertzian mortality (Pollard 1991, 1998; Pollard and Valkovics 1992), and our discussion on the Gompertz model is partly recapitulation and partly elaboration of their work. *M*, $\mu(M)$, *l(M)* and *d(M)* for the Gompertz model are given by:

$$M = \frac{\ln(b/a)}{b} \tag{14.}$$

$$\mu(M) = b \tag{15.}$$

$$l(M) = e^{-1+(a/b)} \approx e^{-1} \tag{16.}$$

$$d(M) = b\, e^{-1+(a/b)} \approx b/e \tag{17.}$$

The expressions for *M* (equation (14)) is implied by $a = be^{-bM}$, which was derived earlier. The expression for $\mu(M)$ (equation (15)) comes from equation (4) by setting *x=M*. The expression for *d(M)* is obtained simply as a product of *l(M)* and $\mu(M)$. Thus, we need to show the derivation of *l(M)* only.

The survival function is given by

$$l(x) = \exp(-\int_0^x \mu(y)dy) \tag{18.}$$

Note that *l(x)* is scaled by *l(0)*=1. Substituting equation (1) into equation (18) and making use of $a = be^{-bM}$, we get

$$l(x) = \exp\left(-\left[\frac{a}{b}e^{by}\right]_{y=0}^{y=x}\right) = \exp(\frac{a}{b} - e^{b(x-M)}), \tag{19.}$$

which leads to $l(M) = e^{-1+(a/b)}$ because usually $a \ll b$. (Note that if the integral in (18) is from -∞ to *x*, *l(M)* will be exactly equal to e^{-1}.)

Logistic model

M, $\mu(M)$, *l(M)* and *d(M)* for the three-parameter logistic model are expressed as:

$$M = \frac{\ln(b/a)}{b} \tag{20.}$$

$$\mu(M) = \frac{b}{1+(b/g)} \approx b \tag{21.}$$

$$l(M) = \left\{ \frac{1+(b/g)}{1+(a/g)} \right\}^{-(g/b)} \approx \{1+(b/g)\}^{-(g/b)} \approx e^{-1} \tag{22.}$$

$$d(M) = \frac{b}{\{1+(b/g)\}^{1+(g/b)}\{1+(a/g)\}^{(g/b)}} \approx \frac{b}{\{1+(b/g)\}^{1+(g/b)}} \approx b/e \tag{23.}$$

It is interesting to note that the expression of M is same for the Gompertz and logistic models. (However, different values of M, a and b will be estimated by fitting these two models to the same data.) As in the case of the Gompertz model, derivations of M, $\mu(M)$, and $d(M)$ are very simple. As for $l(M)$, by substituting of equation (2) into equation (18) and using $a = be^{-bM}$, we have

$$\begin{aligned} l(x) &= \exp\left(-\left[\ln\left\{(g+ae^{by})^{(g/b)}\right\}\right]_{y=0}^{y=x}\right) = \left\{ \frac{g+a\,e^{bx}}{g+a} \right\}^{-(g/b)} \\ &= \left\{ \frac{1+(b/g)\,e^{b(x-M)}}{1+(a/g)} \right\}^{-(g/b)} \end{aligned} \tag{24.}$$

By setting $x=M$, we have $l(M) = [\{1+(b/g)\}/\{1+(a/g)\}]^{-(g/b)}$. Because a/g is very small, this is close to $1/\{1+(b/g)\}^{(g/b)}$. The denominator of the ratio converges to e as b/g approaches zero. Because b/g is fairly small, $l(M)$ is expected to be close to e^{-1}. (It helps to have some idea about plausible ranges of parameter ratios, a/b, b/g and a/g. Note that the three parameters (a, b and g) are approximately or exactly the forces of mortality at different ages: a is nearly equal to extrapolated $\mu(0)$, b is close to actual $\mu(M)$, and g is the upper limit of μ. With typical rounded values such as M=80, b=0.1 and g=1, we have a/b=0.0003, b/g=0.1 and a/g=0.00003.)

<u>Weibull model</u>

Expressions for M, $\mu(M)$, $l(M)$ and $d(M)$ in the Weibull model are shown below:

$$M = (b/a)^{1/(b+1)} \tag{25.}$$

$$\mu(M) = b/M \tag{26.}$$

$$l(M) = e^{-b/(b+1)} \approx e^{-1} \tag{27.}$$

$$d(M) = \left(b / M\right) e^{-b/(b+1)} \approx b/(e\,M) \qquad (28.)$$

The expression for M is implied by $a = b\,M^{-(b+1)}$, which was shown earlier. The expression for *μ(x)* is obtained by simply setting of x=M in equation (6). As for *l(M),* substitution of equation (3) into equation (18) results in

$$l(x) = \exp\left(-\left[\frac{a}{b+1}y^{b+1}\right]_{y=0}^{y=x}\right) = \exp\left\{-\left(\frac{a}{b+1}\right)x^{b+1}\right\} = \exp\left\{-\left(\frac{b}{b+1}\right)\left(\frac{x}{M}\right)^{b+1}\right\}, \qquad (29.)$$

so that $l(M) = e^{-b/(b+1)}$, which is slightly higher than e^{-1} because the value of b in the Weibull model is typically 6~10.Then, the expression for *d(x)* follows.

<u>Comparison among the models</u>

It should be noted that the expressions of M-related measures for the three models are fairly similar. Each of the three expressions for M (equations (14), (20) and (25)) includes *b/a*, suggesting that M is determined by the relative difference between the rate of age-related mortality increase (which is the rate of exponential, logistic, or polynomial increase) and the force of mortality at the reference age (0 for Gompertz and logistic, and 1 for Weibull).

As indicated by equations (16), (22) and (27), *l(M)* in each of the three models is close to (though slightly larger than) $e^{-1} \approx 0.368$. Interestingly, even though the overall level of adult mortality changes substantially, *l(M)/l(30)* is expected to be fairly constant (assuming mortality above age 30 is well approximated by those models). It can also be expected that as mortality at young ages continues to decline, l(30) moves toward unity and l(M)/l(30) comes closer to l(M), so that the age at which l(x)=e^{-1} approaches M. This prediction is supported by the data shown in figure 27.

The expressions for *μ(M)* (equations (15), (21) and (26)) indicate that a rise in *μ(M)* is associated with a higher value of b, i.e., a steeper increase of age-related mortality increase. Because *l(M)* is fairly stable and close to e^{-1}, an increasing b is also associated with a rise in *d(M)*, which suggests concentration of more deaths in a narrower age range around M.

3. Modeling post-modal death distribution

Lexis model and its characteristics

Although all of the models are presented in terms of $\mu(x)$, the model by Lexis is formulated with respect to *d(x)*. As described previously, a main feature of the Lexis model is the normal distribution of *d(x)* above age M:

$$d(x) = l(M)\frac{2}{s\sqrt{2\pi}}\exp\left(-\frac{(x-M)^2}{2s^2}\right) \quad \text{for any} \quad x \geq M \;, \qquad (30.)$$

where s is the standard deviation of the normal distribution. s is SD(+) in earlier sections of this paper, but we use different notations for the parameter of the normal distribution and the quantity computed from empirical data.

Because of the assumption of normal distribution, *s, e(M)* and $\mu(M)$ are related with each other in numerically specific manner:

$$e(M) \approx 0.798s \qquad (31.)$$

$$\mu(M) \approx 0.798/s \qquad (32.)$$

These numerically specific relationships are derived from the assumption of normal distribution. By making use of equation (30), the life expectancy at the modal age at death is given by

$$\begin{aligned} e(M) &= \frac{\int_M^\infty (x-M)d(x)dx}{l(M)} \\ &= \sqrt{\frac{2s^2}{\pi}}\int_M^\infty \frac{x-M}{s^2}\exp\left(-\frac{(x-M)^2}{2s^2}\right)dx \\ &= \sqrt{\frac{2s^2}{\pi}}\left[-\exp\left(-\frac{(x-M)^2}{2s^2}\right)\right]_M^\infty \\ &= \sqrt{\frac{2s^2}{\pi}} \approx 0.798\, s \end{aligned}$$

The force of mortality at M is obtained in the following way. The survival function for any age above M is given by

$$l(x) = \int_x^\infty d(y)dy$$
$$= \int_x^\infty l(M)\frac{2}{s\sqrt{2\pi}}\exp\left(-\frac{(y-M)^2}{2s^2}\right)dy$$

so that

$$\frac{dl(x)}{dx} = -l(M)\frac{2}{s\sqrt{2\pi}}\exp\left(-\frac{(x-M)^2}{2s^2}\right).$$

Thus,

$$\mu(M) = -\frac{dl(M)/dx}{l(M)} = \frac{2}{s\sqrt{2\pi}} = \frac{\sqrt{2/\pi}}{s} \approx \frac{0.798}{s}.$$

Comparison of Lexis and logistic models

The Lexis model fit data above M very well (Cheung and Robine 2006). On the other hand, logistic models have been shown to fit old-age mortality data remarkably well (Thatcher, Kannisto, and Vaupel 1998; Thatcher 1999). Thus it seems important to investigate relationships between the Lexis model, expressed in terms of *d(x)*, and logistic model, expressed in terms of $\mu(x)$.

We fitted both models to some empirical mortality schedules and found that not only both of the models fit data well, but also the mortality schedules estimated by the two models are nearly identical. Figure 28 shows the results for Japanese females, 1995-1999, as an example. In what follows, we will explore the reason why the two models could produce almost same patterns of *d(x)* for post-modal ages.

In order to investigate *d(x)* patterns in detail, it is useful to focus on the rate of relative decrease in d(x), which is defined as

$$v(x) = -d'(x)/d(x) = \mu(x) - k(x) \quad . \qquad (33.)$$

v(x) is expected to rise with age after *M*.

The Lexis model for post-modal ages is expressed as:

$$d(x) = l(M)\frac{2}{s\sqrt{2\pi}}\exp\left(-\frac{(x-M)^2}{2s^2}\right) \quad \text{for any} \quad x \ge M, \qquad (34.)$$

where s is the standard deviation of the (right half of) normal distribution. Using equation (33), we get the *v(x)* function for the Lexis model:

$$v(x) = -\frac{d'(x)}{d(x)} = \frac{1}{s^2}(x - M). \qquad (35.)$$

Thus, the rate of relative decrease in *d(x)* for the Lexis model rises linearly with age, and the slope of the linear rise is $1/s^2$.

The *v(x)* function for the logistic model is obtained by substituting equations (2) and (12) into equation (33):

$$v(x) = \frac{a\,e^{bx} - b}{1 + (a/g)\,e^{bx}} = \frac{b(\,e^{b(x-M)} - 1)}{1 + (b/g)\,e^{b(x-M)}} \quad . \qquad (36.)$$

This is another logistic curve with the increase rate *b*, but unlike the logistic function of $\mu(x)$ (equation (2) or (12)), it is bounded by –*b* and *g*. It is useful to know that the point of inflection of this *v(x)* curve can be found at age Z, which is given by

$$Z = \frac{\ln(g/a)}{b} = M + \frac{\ln(g/b)}{b} \quad . \qquad (37.)$$

This is obtained by setting $v''(Z) = 0$ and comparing the result with equation (20). *Z* is older than *M* because $g>b>0$.

Thus the mathematical forms of v(x) for the Lexis and logistic models are clearly different: linear versus logistic functions of age. However, it should be noted that a logistic curve has three parts that are relatively straight: the fairly flat part near the lower bound, the steeply rising part around the point of inflection, and the fairly flat part near the upper bound. Substantial curvatures tend to be found in transitory parts between these relatively straight parts (see figure 29).

Equation (34) implies that the age range of [M, 2Z-M], which centers at the age of inflection point *Z*, covers *2ln(g/b)/b* years. Figure 29 shows that this age range is from 89.4 to 112.0 for Japanese females in 1995-1999, and that the segment of *v(x)* curve in this range is fairly straight. A straight line can be drawn to pass through the two end points, indicated by asterisks, and the inflection point, marked by "+", and the slope of the line will be *b(g-b)/2(ln(g)-ln(b))*.

Therefore, if parameters *b* and *g* in the logistic model and *s* in the Lexis model are set up to satisfy

$$\frac{b\,(g-b)}{2\,(\ln(g)-\ln(b))} \approx \frac{1}{s^2} \quad , \qquad (38.)$$

and if the *v(x)* curve for the logistic model is fairly straight in the age range [M, 2Z-M], the two *v(x)* functions should be close to each other, leading to similar *d(x)* functions for the age range. Note that $d(x) = d(M)\exp(-\int_M^x v(y)dy)$, i.e., the d-function is directly related to the integral of v-function, and thus two *fairly similar* sequences of *v(x)* values are likely to produce two *very similar* sequences of *d(x)* series. As seen in figure 29, the curve segment might not be very straight if *M* is far from *Z*. However, with empirical data, the distance between *M* and *Z* is limited by plausible numerical ranges of *b* and *g* in logistic models fitted to observed life tables. Thus, the Lexis model and logistic model, if fitted to the same data, are likely to produce numerically almost identical d(x) distributions for post-modal ages.

It should be noted, however, that if the Lexis and logistic models are extrapolated to extremely high ages, they depart from each other: the *v(x)* function for the logistic model converges to *g*, but that for the Lexis model keeps increasing linearly. The departure is obvious with respect to the force of mortality. $\mu(x)$ in the logistic model converges to the upper limit, whereas $\mu(x)$ in the Lexis model, expressed as

$$\exp\left(-\frac{(x-M)^2}{2s^2}\right)dy \Bigg/ \int_x^\infty \exp\left(-\frac{(y-M)^2}{2s^2}\right)dy$$

continues to rise with age.

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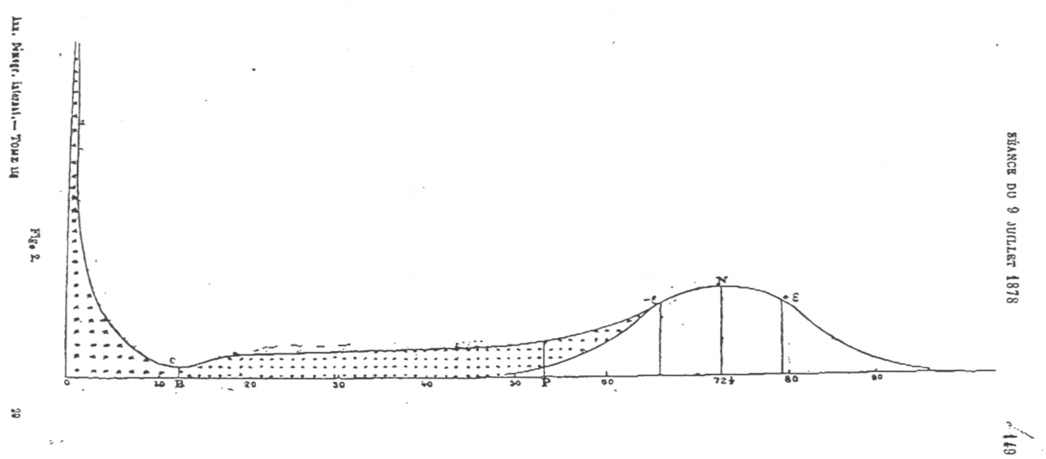
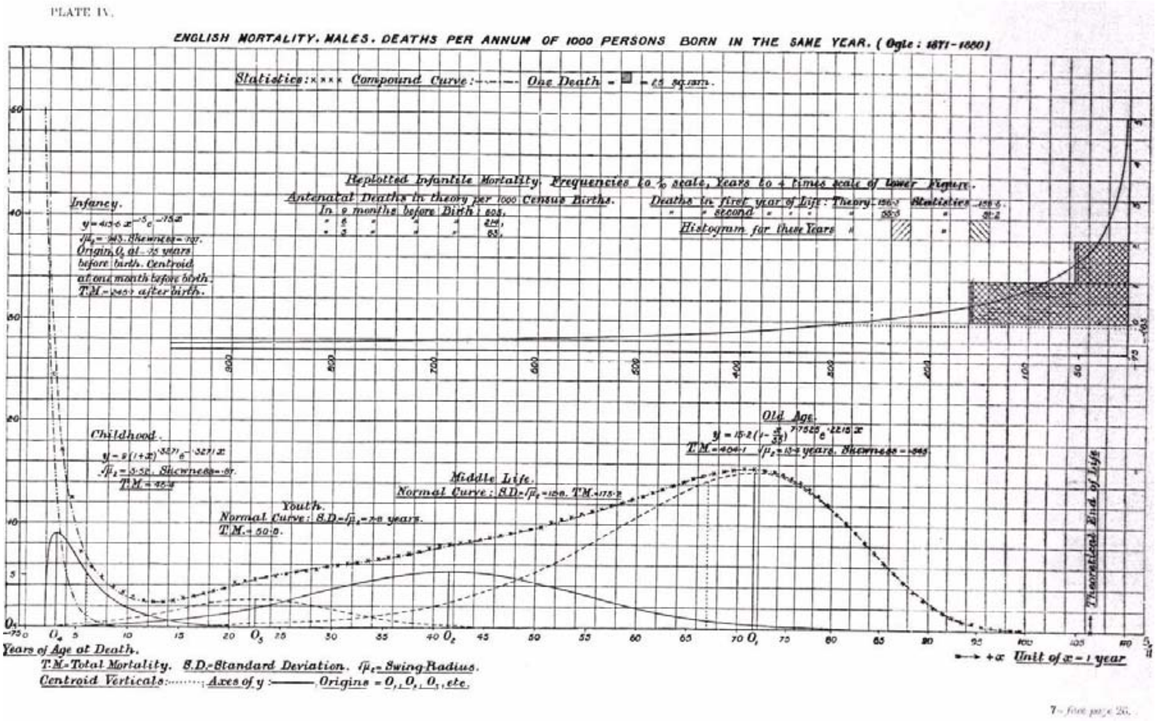
FIGURE 1. LEXIS'S NORMAL LIFE DURATIONS



Source: Adapted from Lexis (1878)

FIGURE 2. PEARSON’S FIVE COMPONENTS OF A CHANCE DISTRIBUTION AT DEATH AND THE THEORETICAL END OF LIFE



Source: Adapted from Pearson (1897)

FIGURE 3. BENJAMIN'S THE LATER MODE AND THE RIGHT-HAND SIDE OF THE DISTRIBUTION OF "SENESCENT" DEATHS

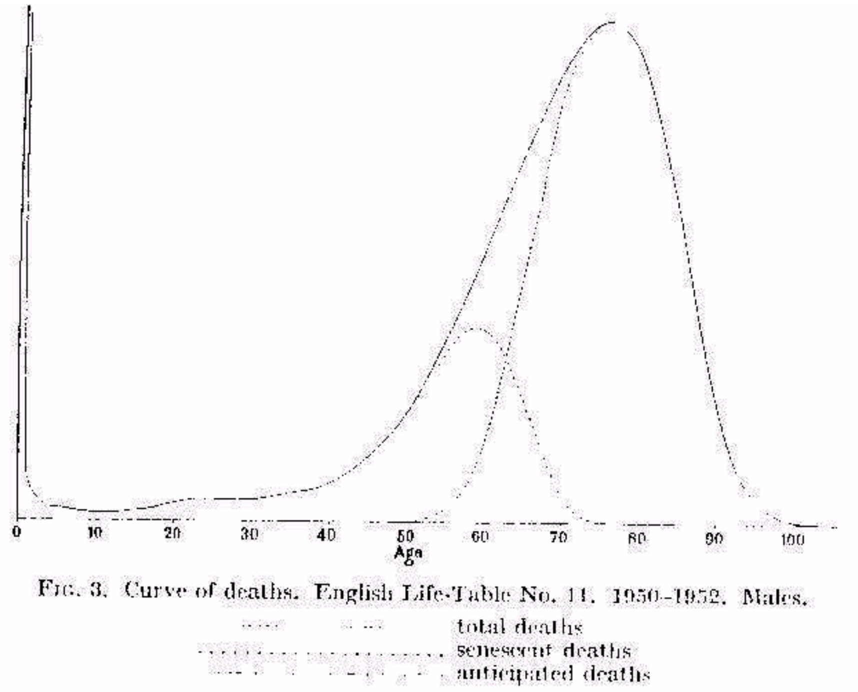


Fig. 3. Curve of deaths. English Life-Table No. 11. 1950–1952. Males.

Source: Adapted from Benjamin (1959)

FIGURE 4. CHANGES IN THE PROPORTION OF NEWBORN REACHING ADULTHOOD AS INDICATED BY lx=18, UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

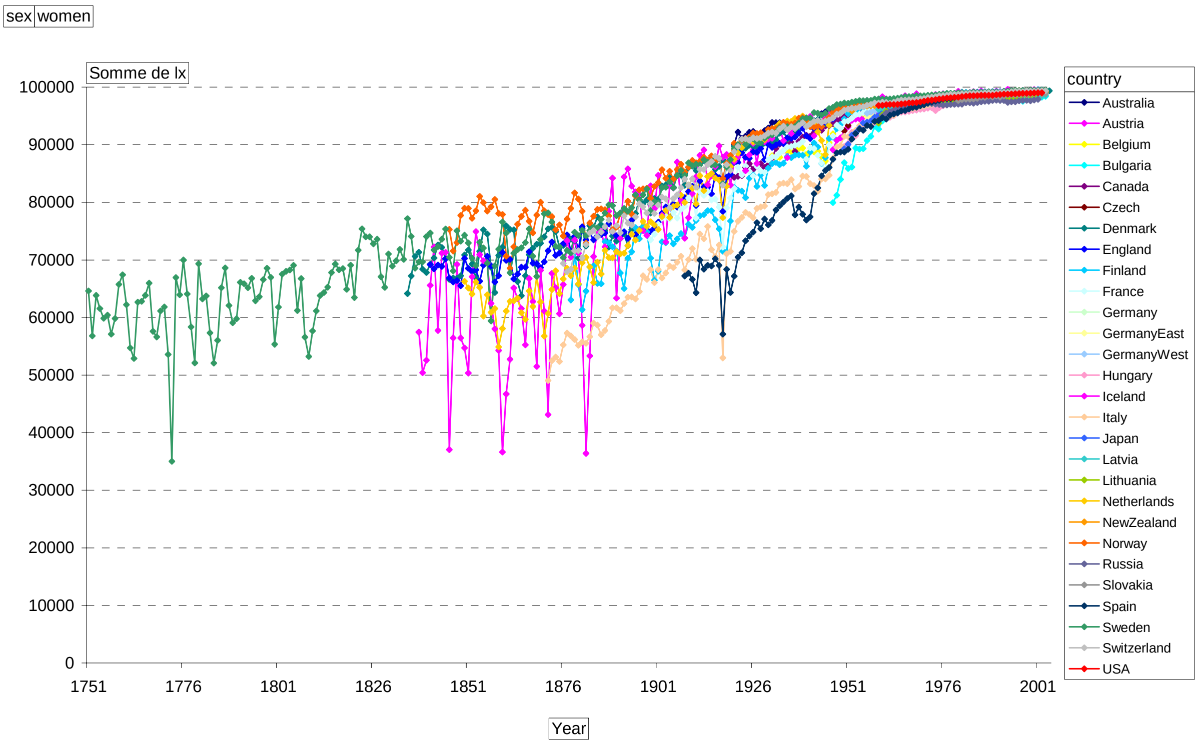


FIGURE 5. CHANGES IN THE MODAL AGE AT DEATH (M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

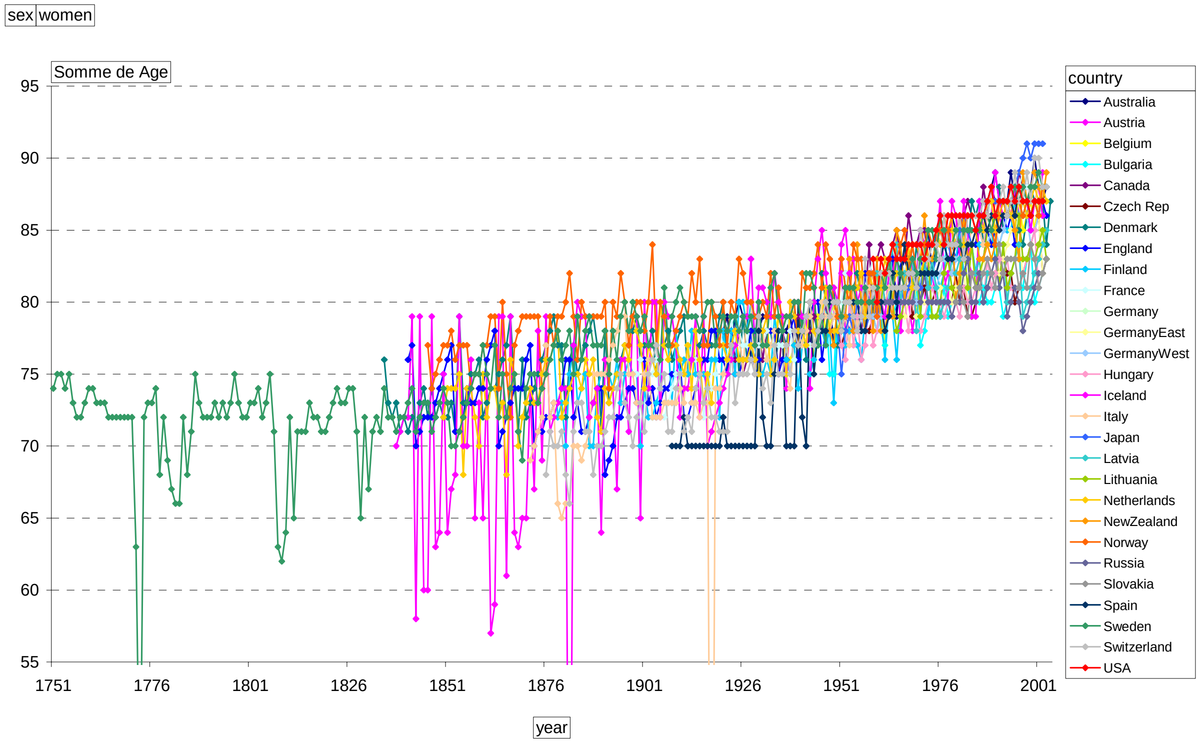


FIGURE 6. CHANGES IN THE STANDARD DEVIATION OF THE AGES AT DEATH ABOVE M (SD(M+)), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

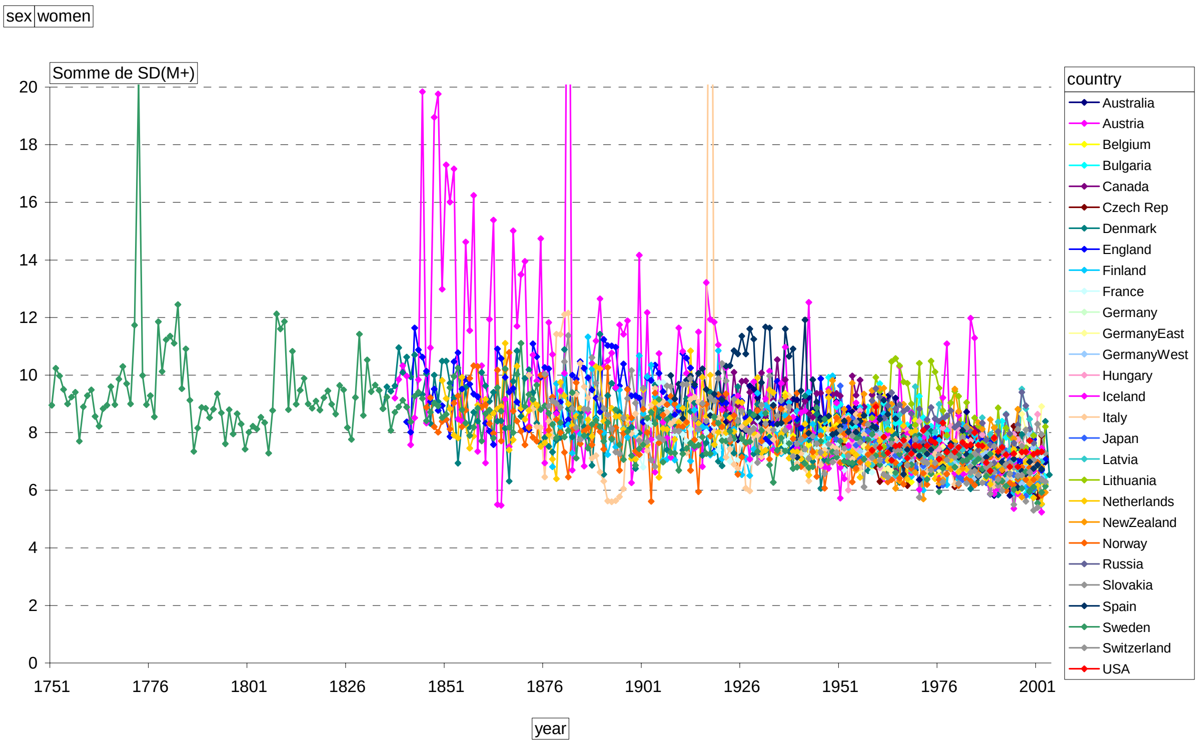


FIGURE 7. CHANGES IN M+3SD(M+), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

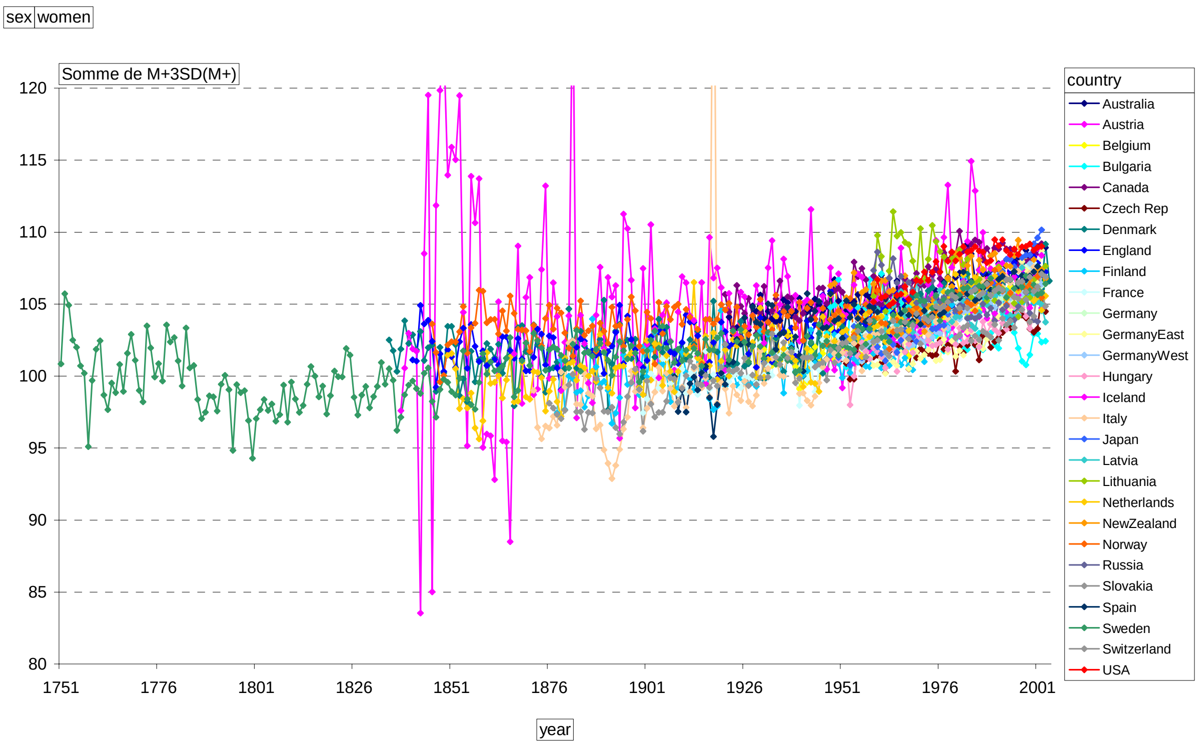


FIGURE 8. CHANGES IN M+4SD(M+), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

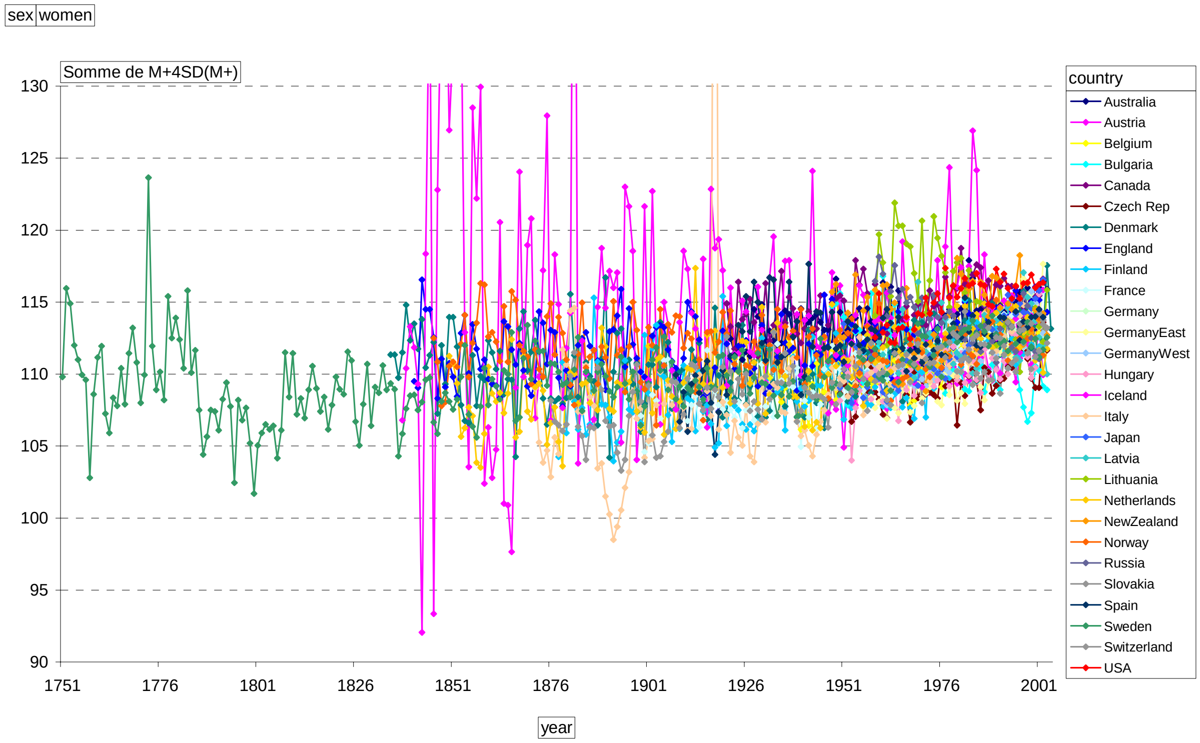


FIGURE 9. CHANGES IN THE PROBABILITY OF DEATH AT M, UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

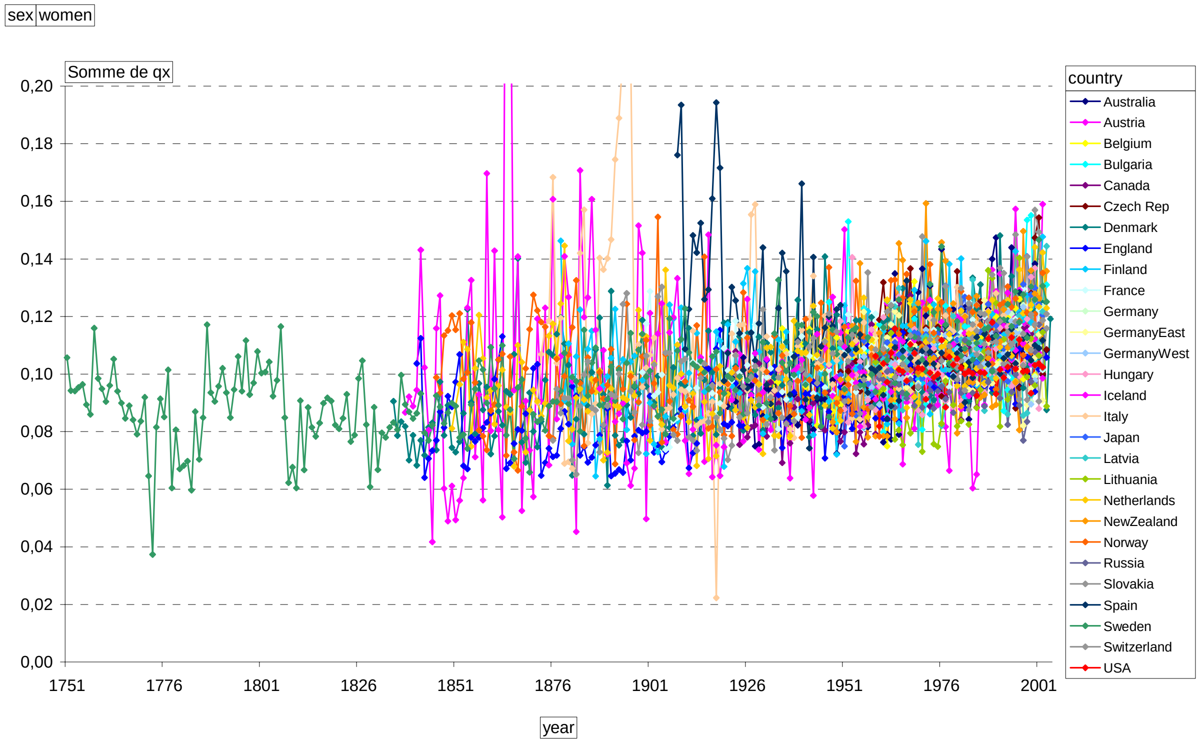


FIGURE 10. CHANGES IN THE NUMBER OF SURVIVORS AT M (lx=M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

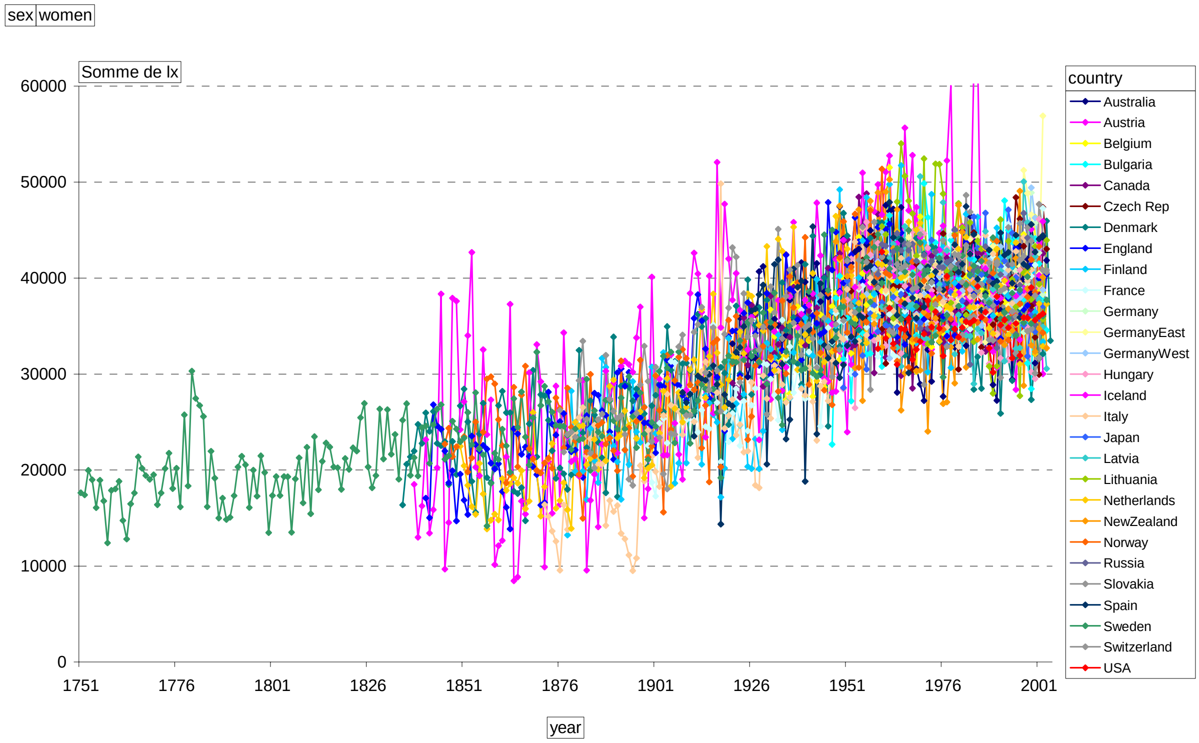


FIGURE 11. CHANGES IN THE NUMBER OF DEATHS OCCURRING AT M (dx=M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

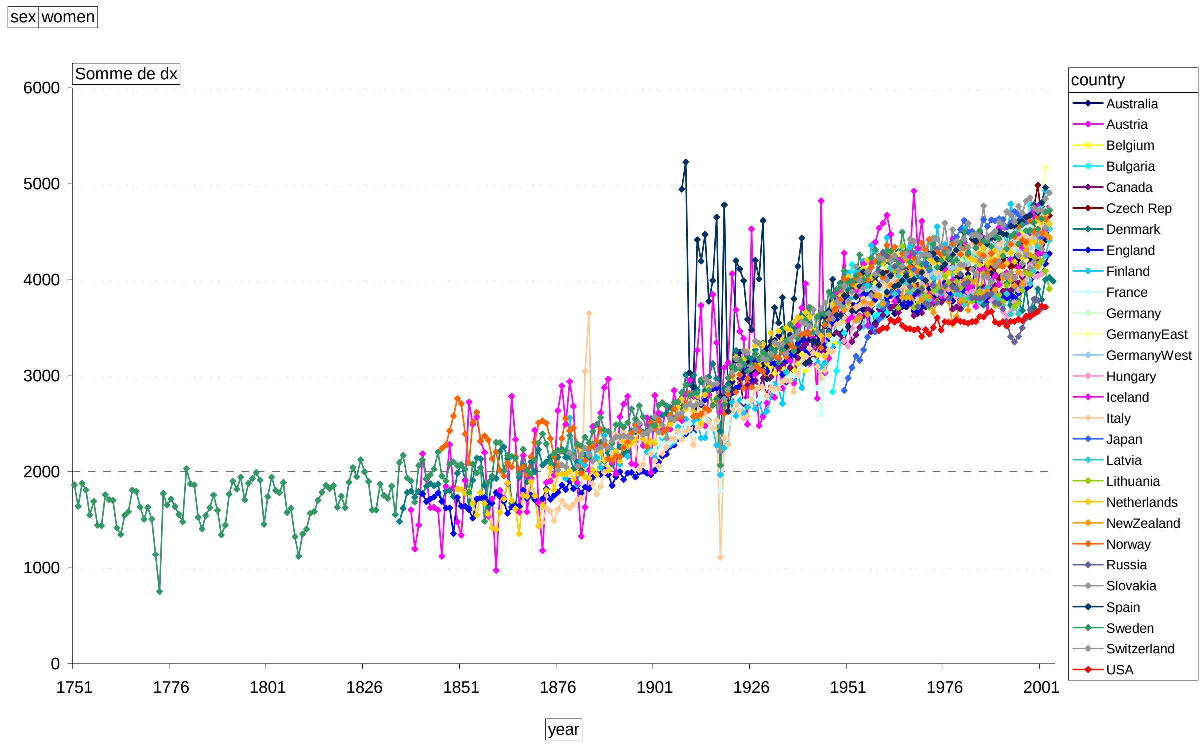
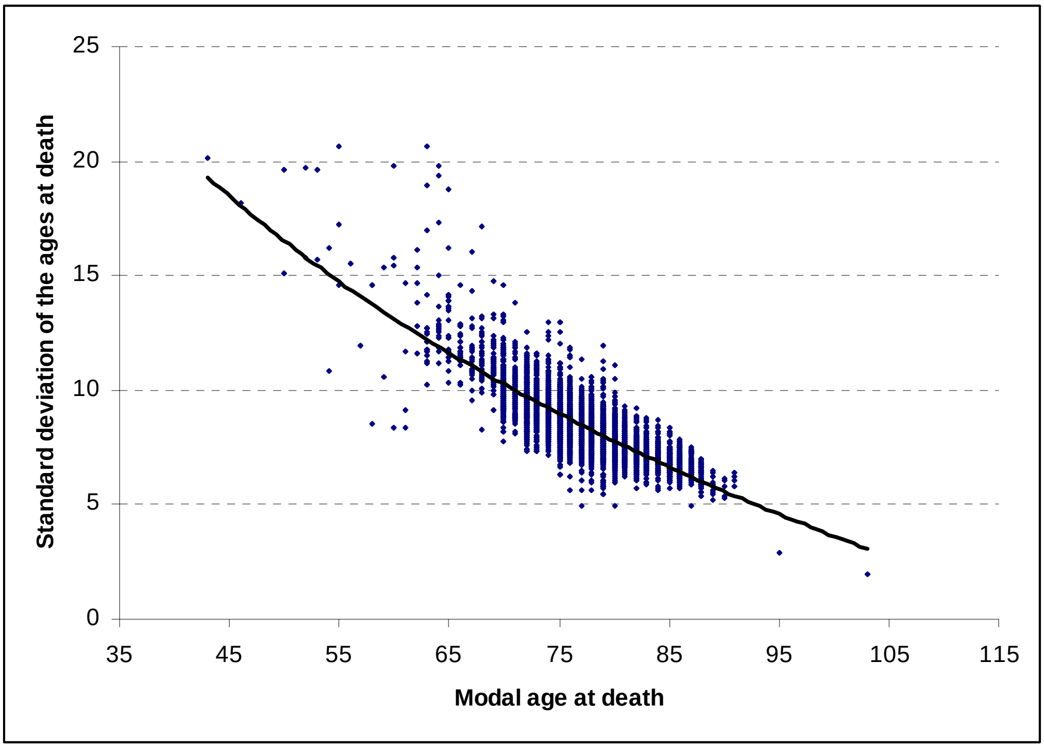


FIGURE 12. CORRELATION BETWEEN THE MODAL AGE AT DEATH (M) AND THE STANDARD DEVIATION OF THE AGES AT DEATH ABOVE M (SD(M+)), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 5003 MALE AND FEMALE LIFE TABLES



*We deleted 29 observations, 27 for M being below 40 years and 2 for SD(M+) being over 25 years

FIGURE 13. CHANGES IN THE MODAL AGE AT DEATH (M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 10 COUNTRIES

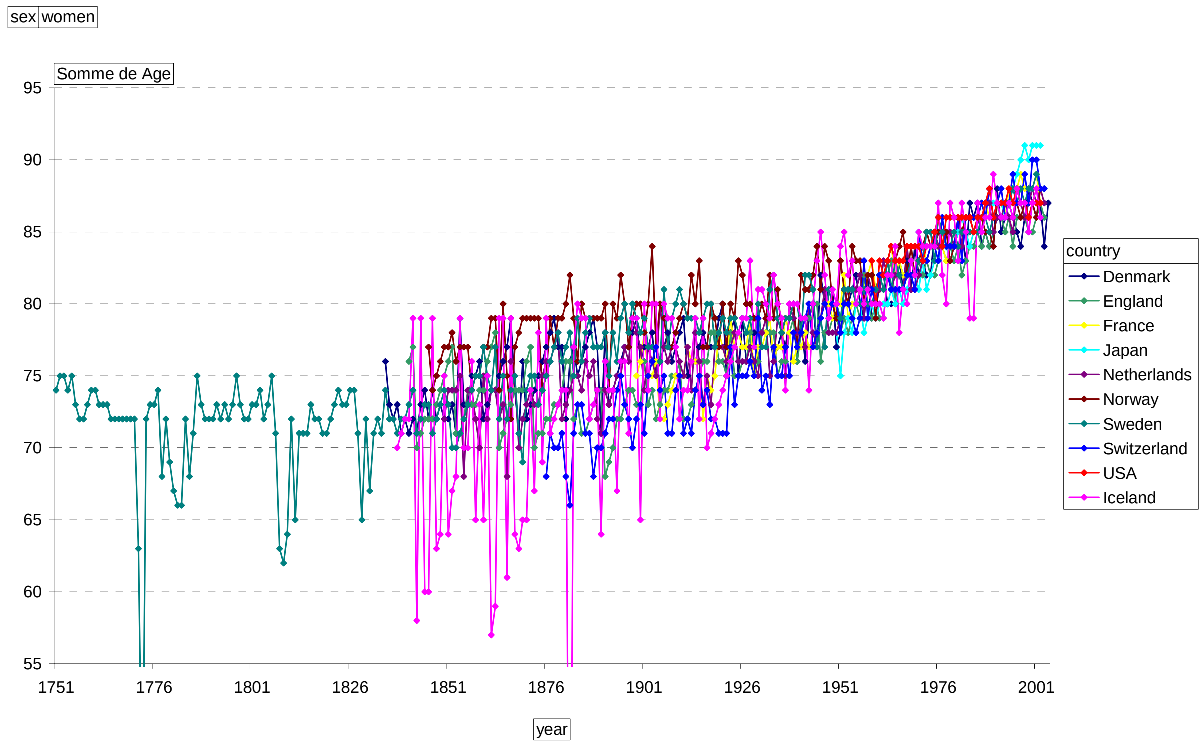


FIGURE 14. CHANGES IN THE MODAL AGE AT DEATH (M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 9 COUNTRIES

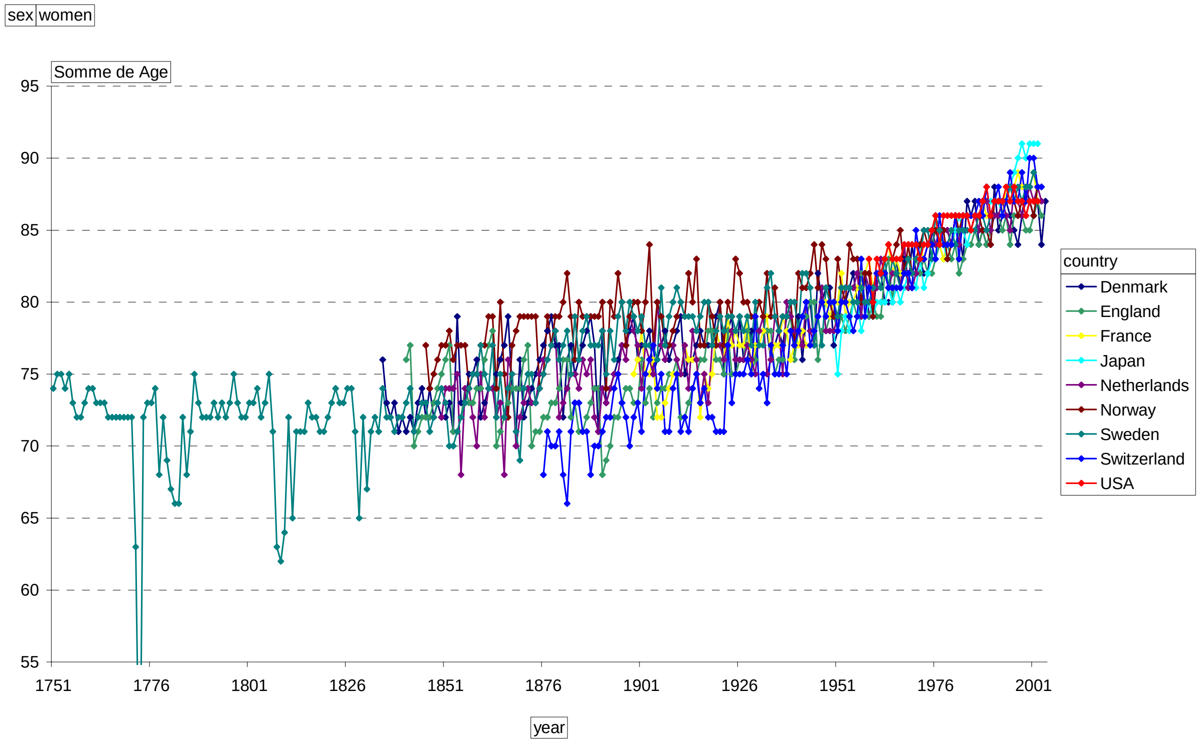


FIGURE 15. CHANGES IN THE STANDARD DEVIATION OF THE AGES AT DEATH ABOVE M (SD(M+)), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 9 COUNTRIES

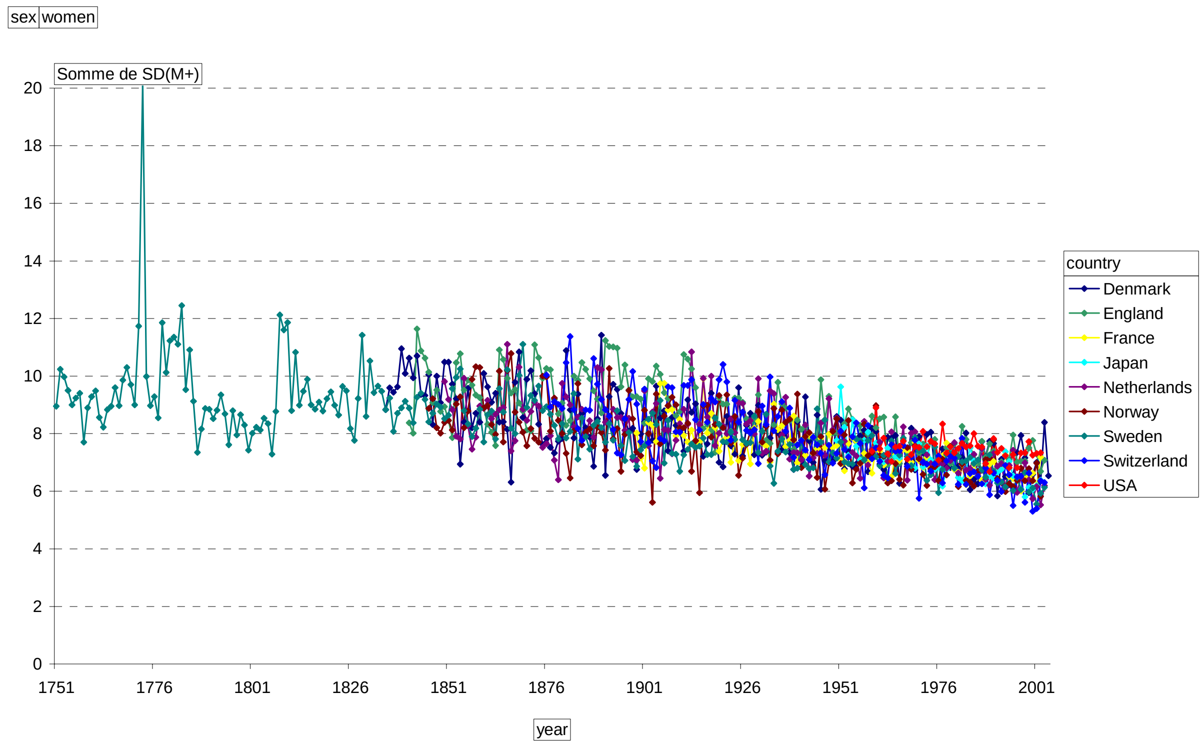


FIGURE 16. CHANGES IN M+3SD(M+), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 9 COUNTRIES

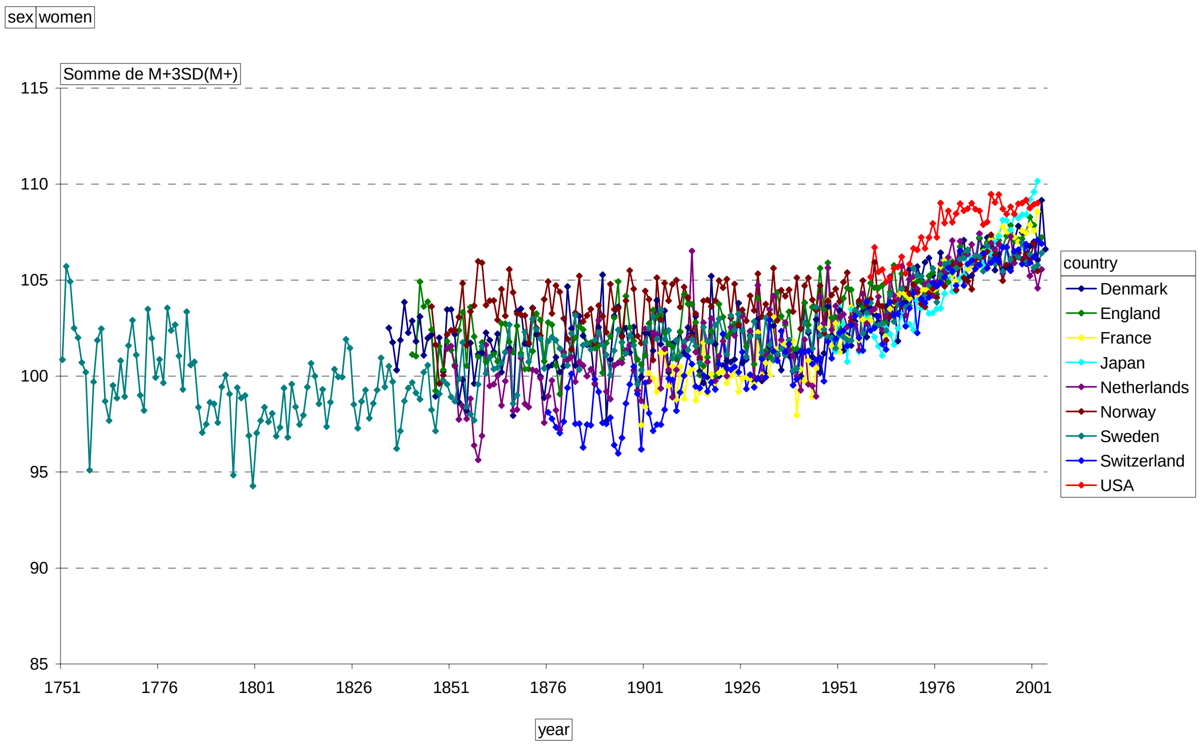


FIGURE 17. CHANGES IN M+3SD(M+), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 8 COUNTRIES

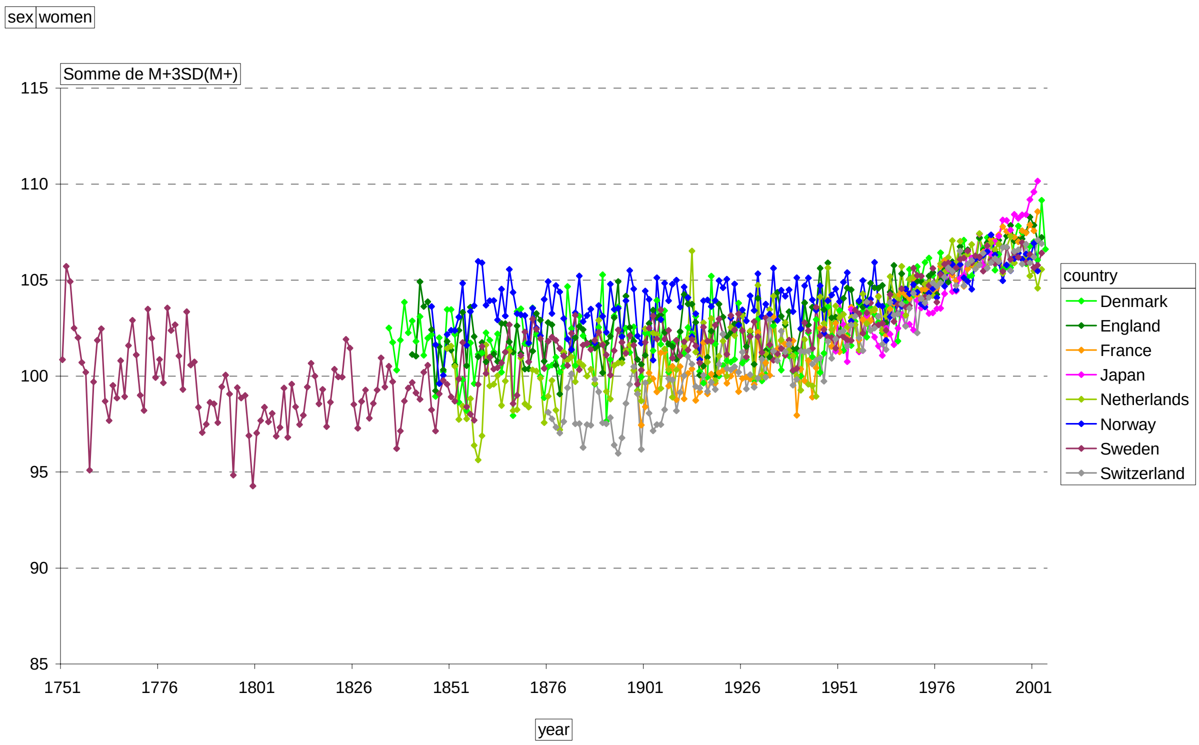


FIGURE 18. CHANGES IN M+3.5SD(M+), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 8 COUNTRIES

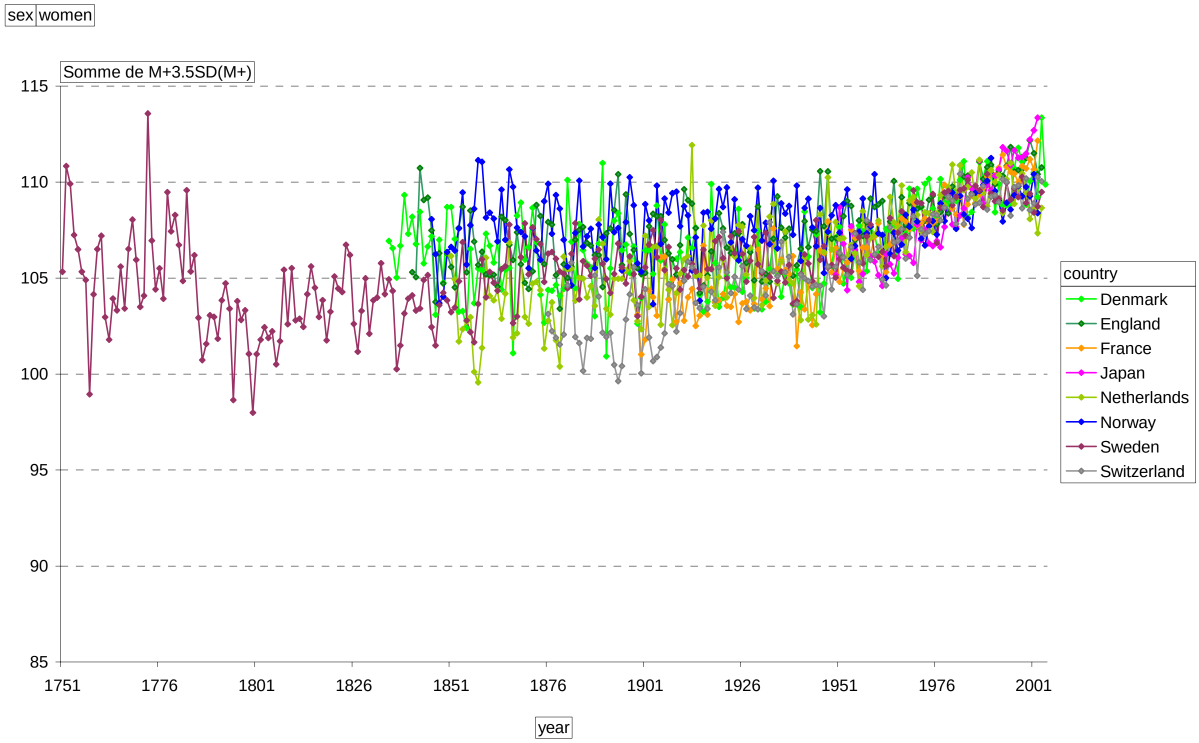


FIGURE 19. CHANGES IN THE PROBABILITY OF DEATH AT M, UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 8 COUNTRIES

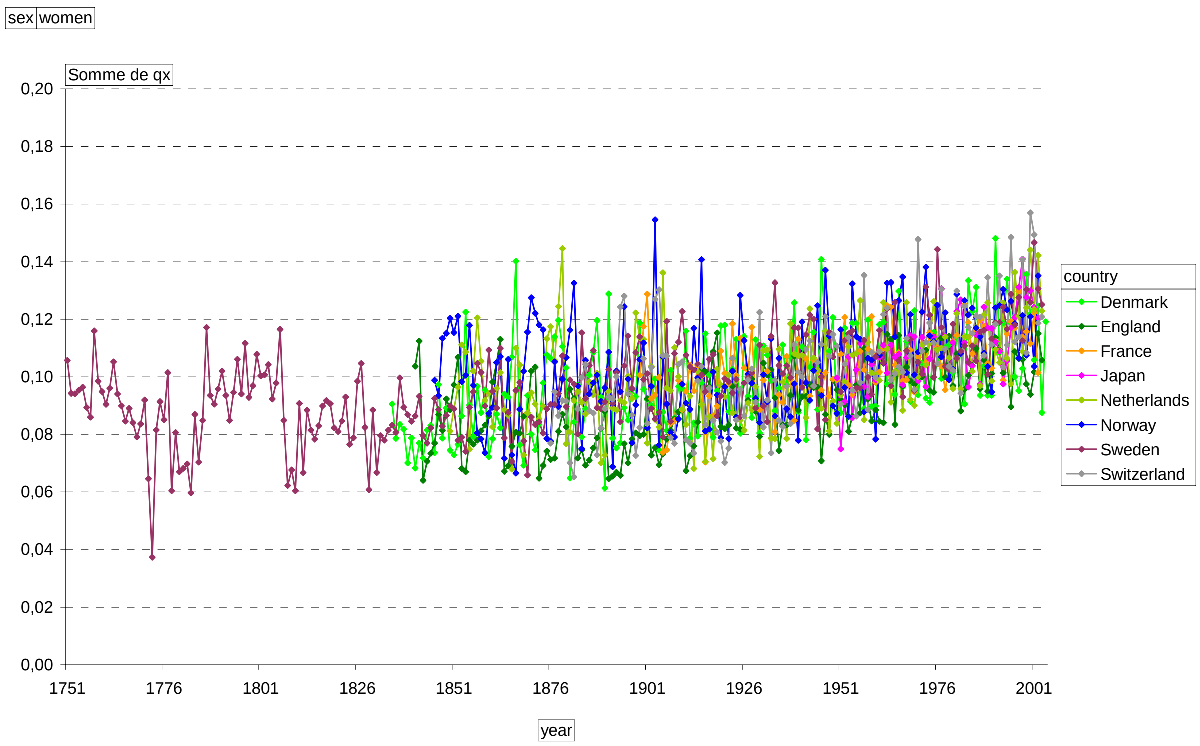


FIGURE 20. CHANGES IN THE NUMBER OF SURVIVORS AT M (lx=M), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: A SELECTION OF 8 COUNTRIES

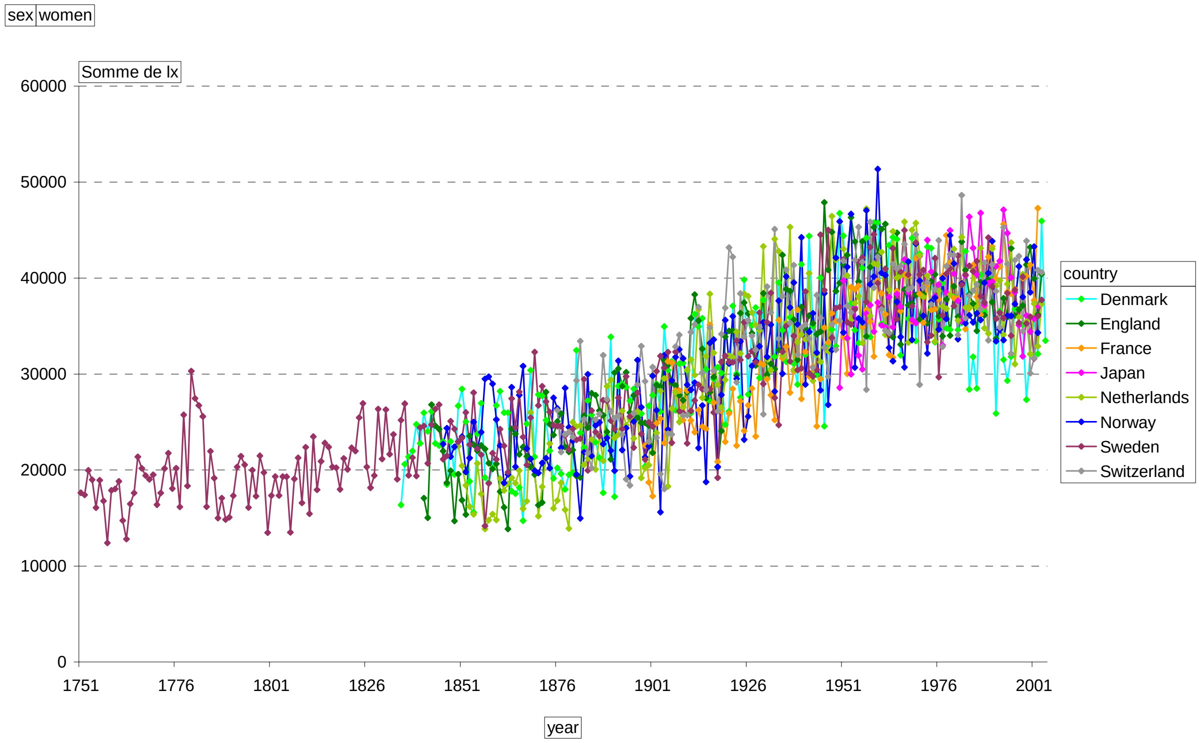
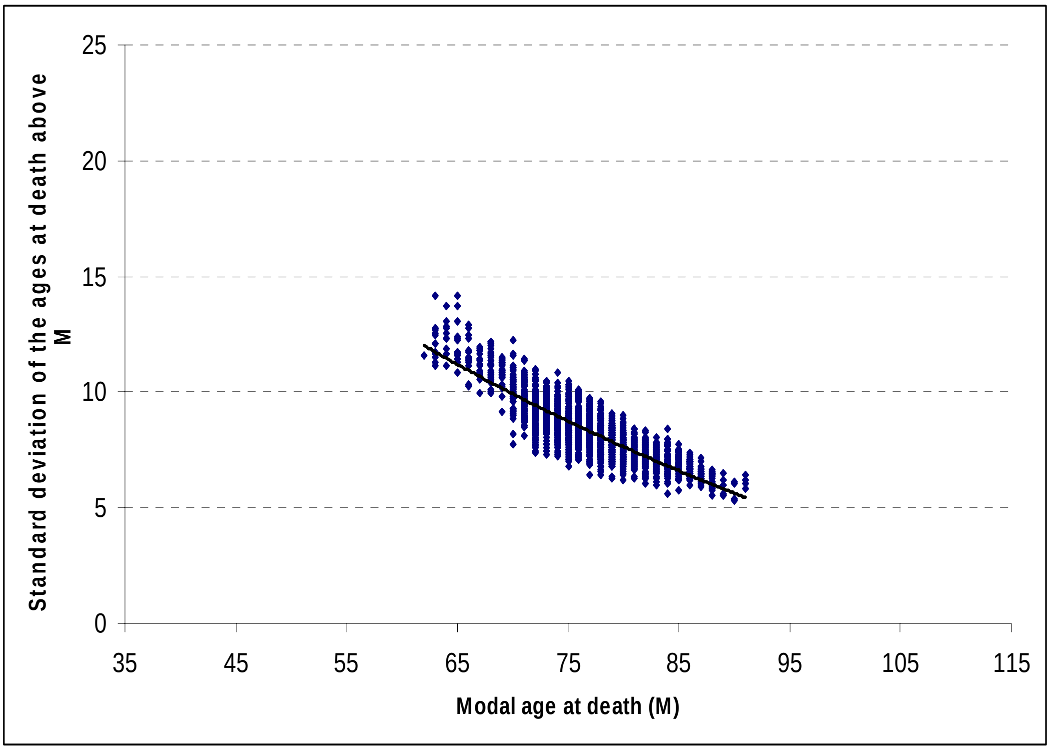


FIGURE 21. CORRELATION BETWEEN THE MODAL AGE AT DEATH (M) AND THE STANDARD DEVIATION OF THE AGES AT DEATH ABOVE M (SD(M+)), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2346 MALE AND FEMALE LIFE TABLES, A SELECTION OF 8 COUNTRIES



***We deleted 18 observations for M being below 50 years**

FIGURE 22. TRENDS IN THE LIFE EXPECTANCY AT BIRTH (DASHED), MEDIAN AGE AT DEATH (DOTTED) AND LATER MODAL AGE AT DEATH (SOLID) FOR FEMALES AND MALES IN FRANCE (1899-2001) AND JAPAN (1950-2002).

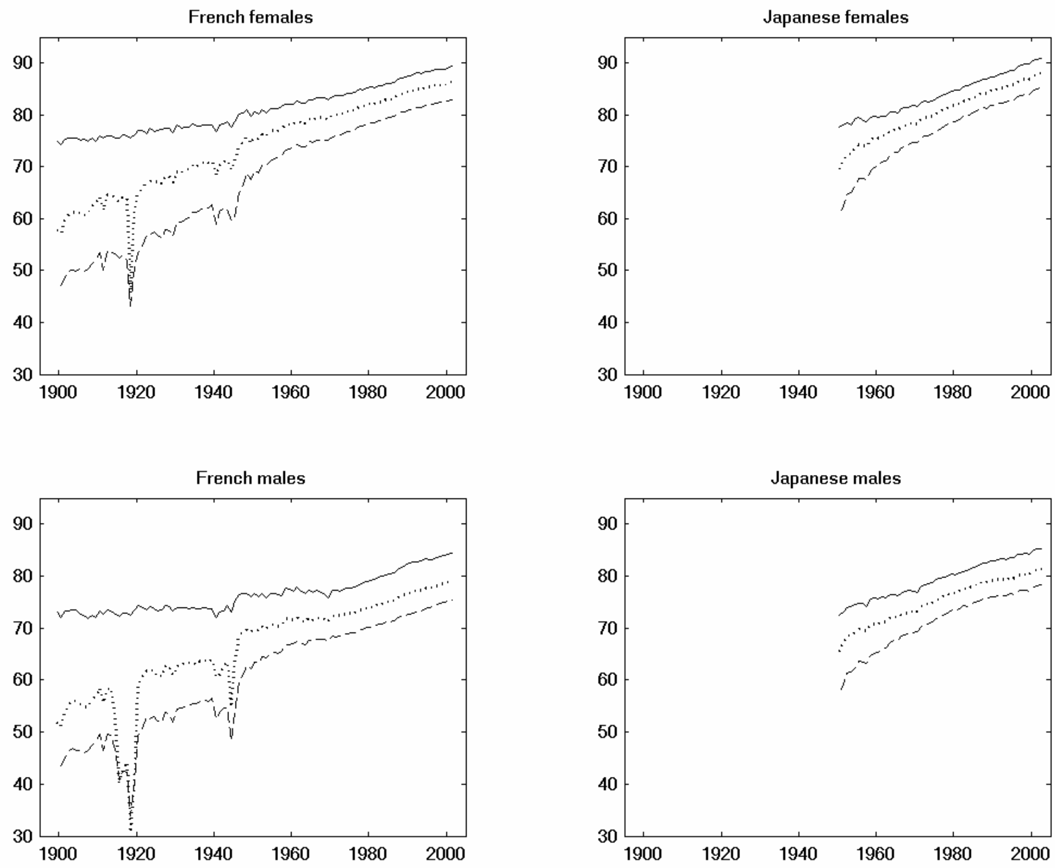
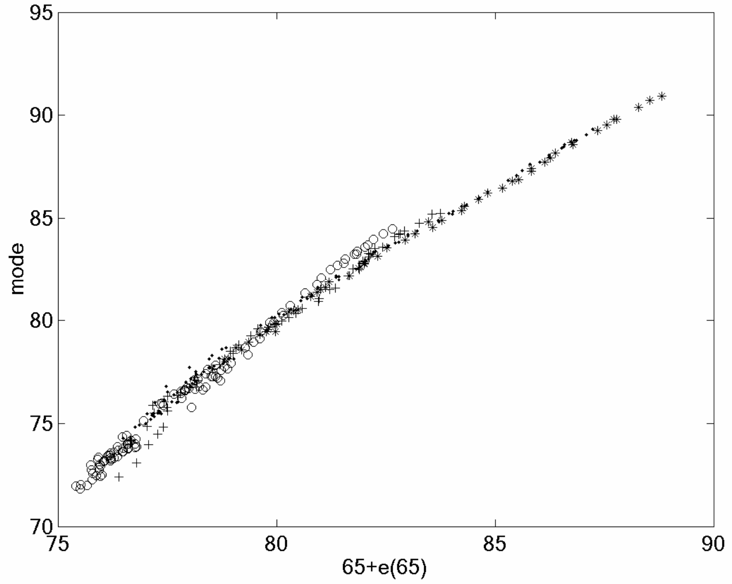


FIGURE 23. RELATIONSHIPS BETWEEN THE LATER MODAL AGE AT DEATH, LIFE EXPECTANCY AT AGE 65, AND LOGARITHM OF GEOMETRIC MEAN OF 5-YEAR AGE-SPECIFIC DEATH RATES BETWEEN 65 AND 99 FOR FRENCH FEMALES (.) AND MALES (O) AND JAPANESE FEMALES (*) AND MALES (+).

(A)



(B)

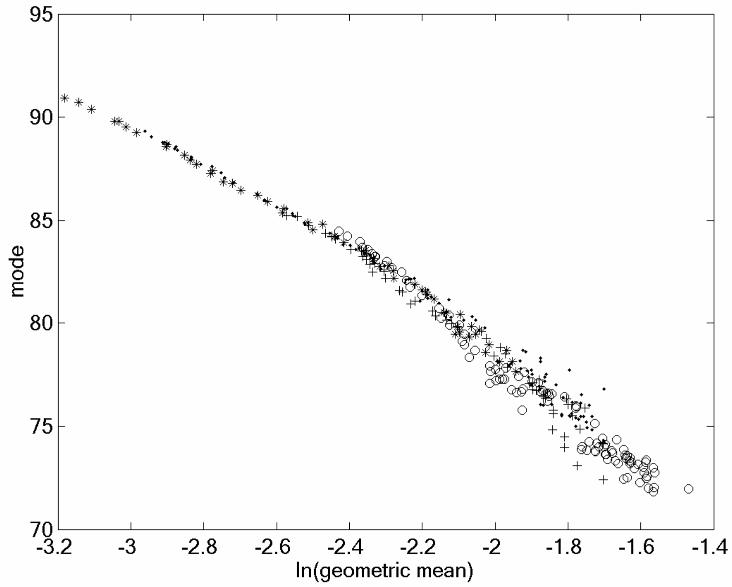


FIGURE 24. CHANGES IN THE AGE AT lx=e(-1), UNDER CURRENT MORTALITY CONDITIONS – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 2516 FEMALE LIFE TABLES

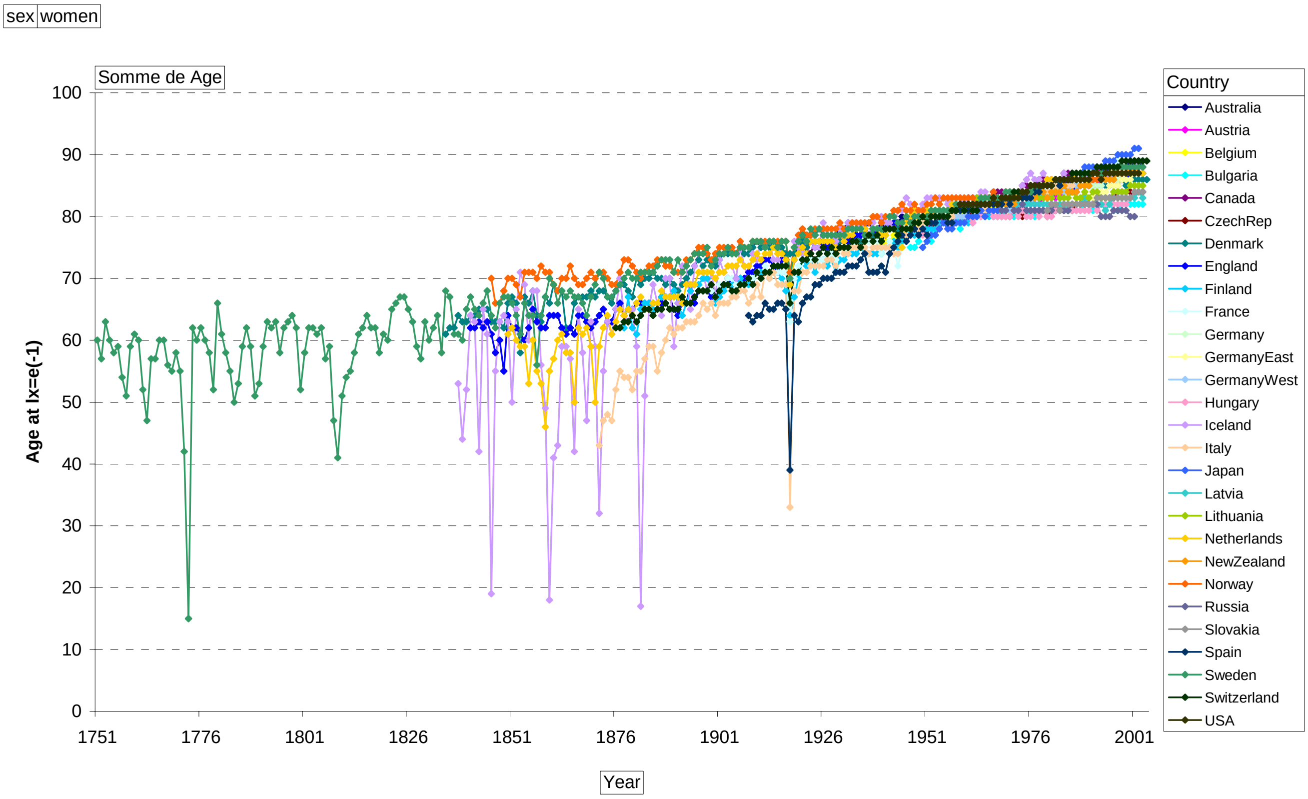


FIGURE 25. FIVE-YEAR AVERAGE MAXIMUM REPORTED AGE AT DEATH (AMRAD) AND M+*k*SD(M+) BY GENDER, ENGLAND AND WALES, JAPAN, SWITZERLAND AND SWEDEN

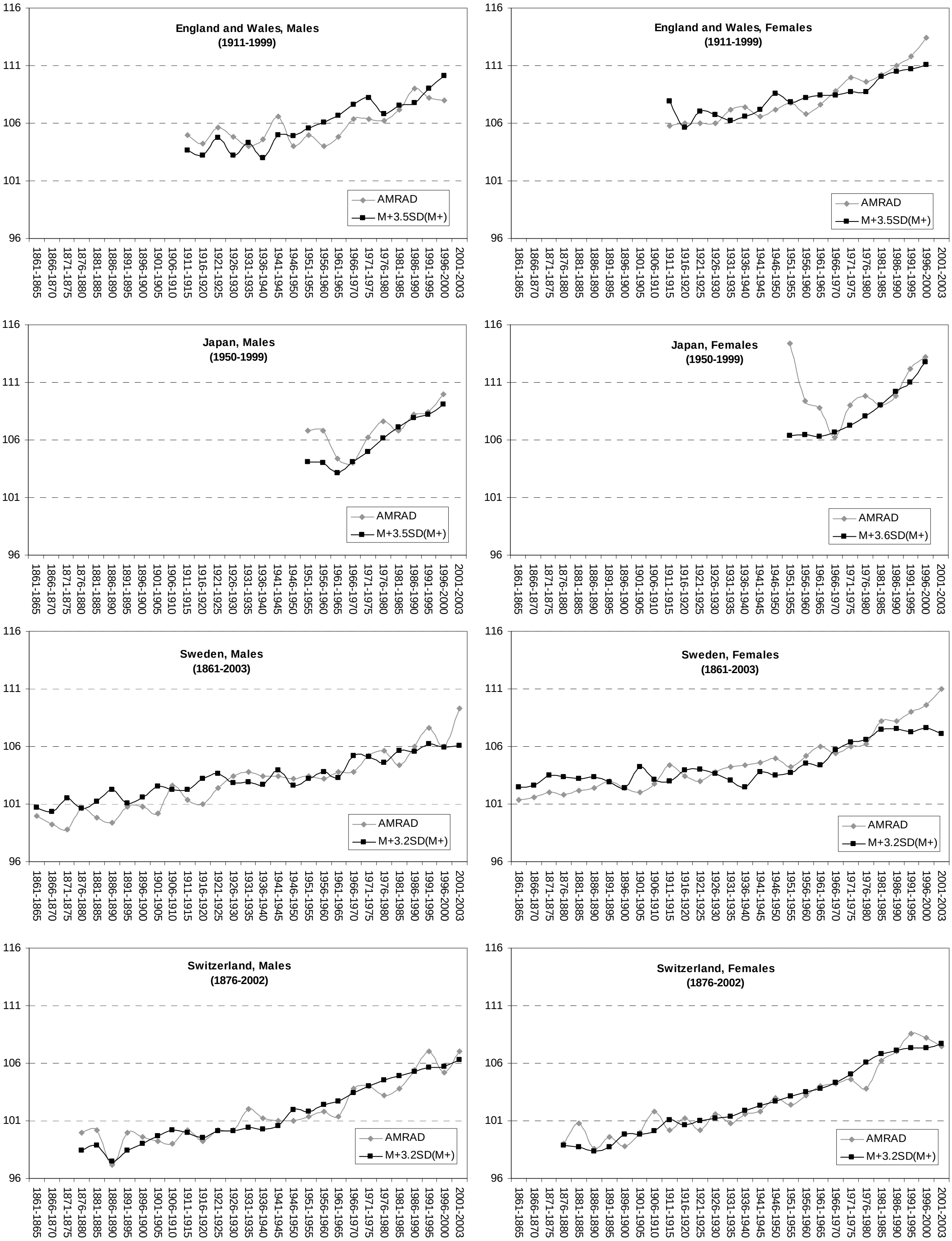


FIGURE 26. CORRELATION BETWEEN THE STANDARD DEVIATION OF THE AGES AT DEATH ABOVE M (SD(M+)) UNDER THE ASSUMPTION OF THE NORMAL MODEL AND THE FORCE OF MORTALITY AT M (μ(M)) – HUMAN MORTALITY DATABASE (HMD), SINCE 1751: 5003 MALE AND FEMALE LIFE TABLES

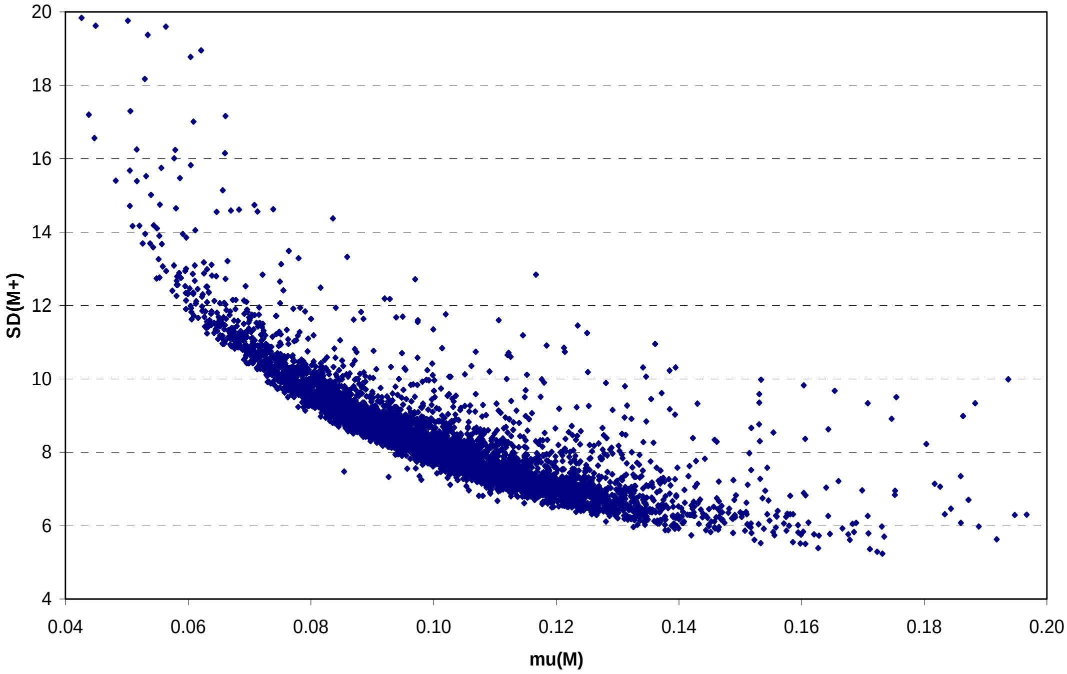


FIGURE 27. CONVERGENCE BETWEEN THE MODAL AGE AT DEATH (M) AND THE AGE AT $l(x)=e^{-1}$ IN ENGLAND & WALES, JAPAN, SWEDEN, AND SWITZERLAND, FEMALES

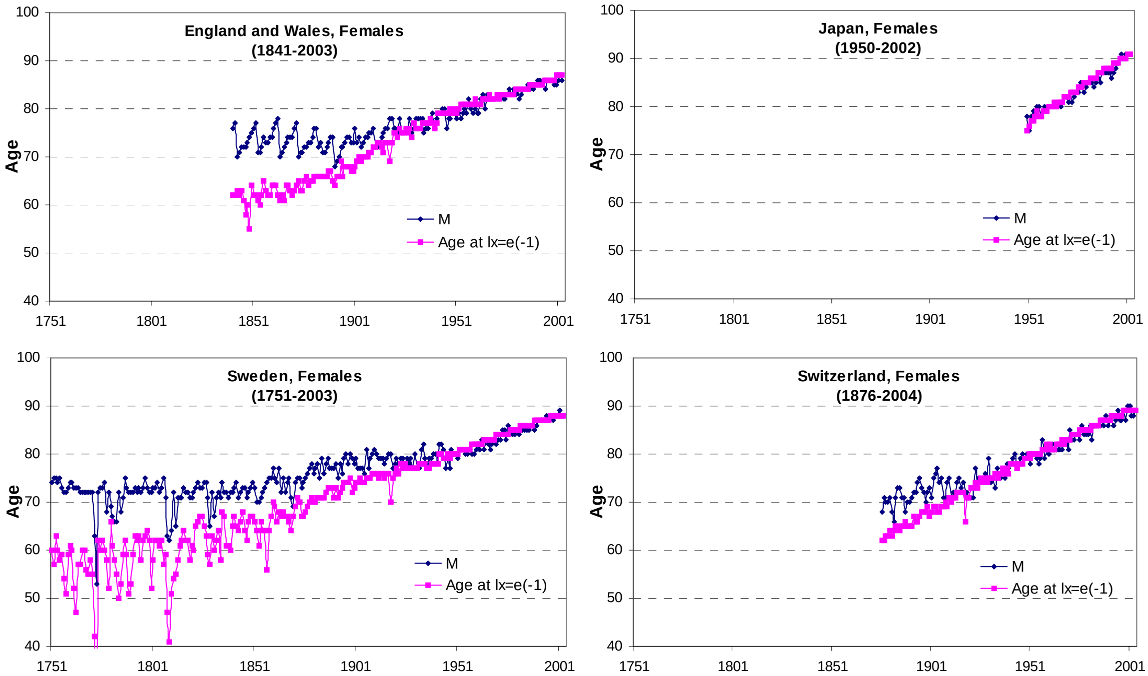
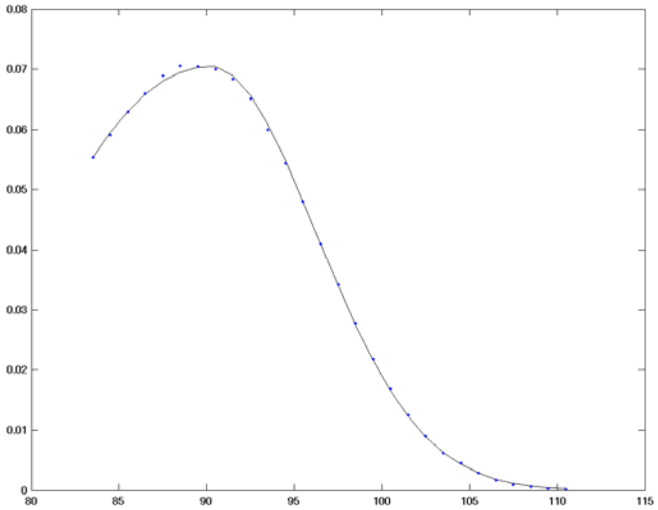
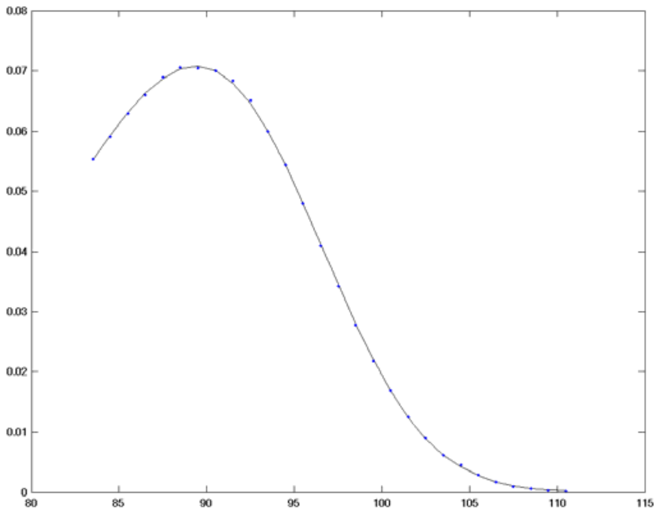


FIGURE 28. THE AGE DISTRIBUTION OF DEATHS FOR JAPANESE FEMALES, 1995-1999: $_1d_x$'S FROM THE PERIOD LIFE TABLE (DOTS) AND ESTIMATES OBTAINED BY FITTING A MODEL TO THE DATA (SOLID LINE).*

(a) Fitting the Lexis model (right half of normal distribution) to life table $_1d_x$'s. **

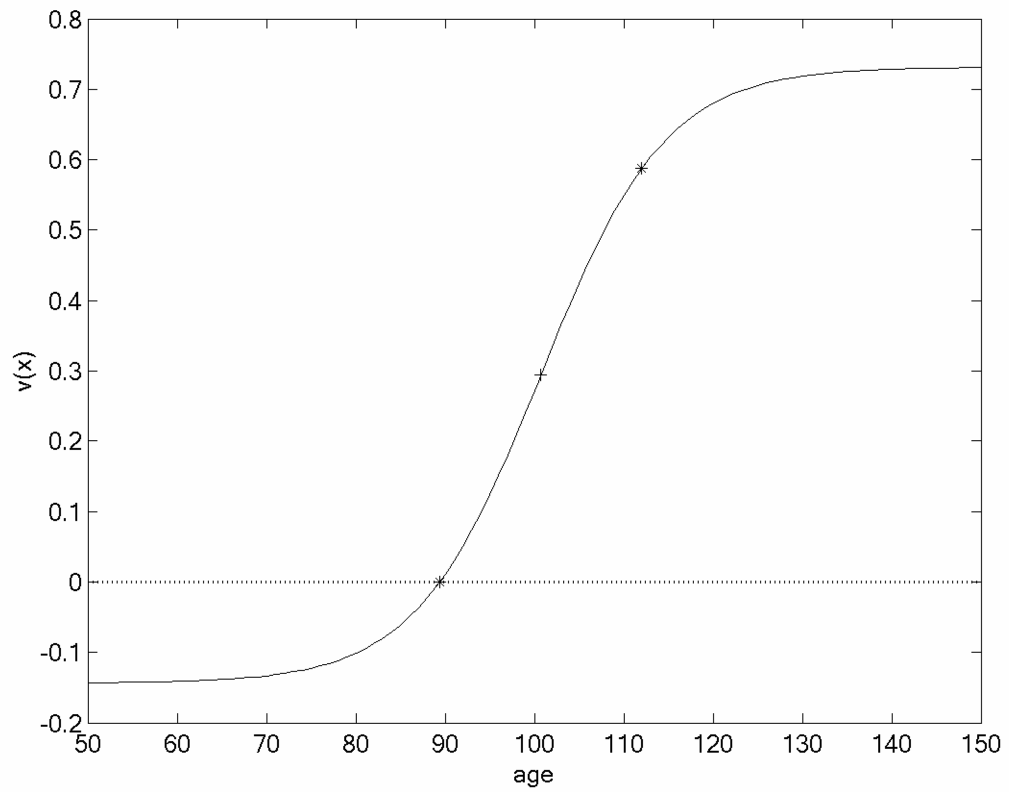


(b) Fitting the three-parameter logistic model to life table $_1d_x$'s.



* Fitted by applying the method of maximum likelihood to life tables as in Thatcher (1999). ** The parabolic function was fitted to pre-modal mortality data.

FIGURE 29. THE RATE OF RELATIVE D(X) DECREASE FOR THE THREE-PARAMETER LOGISTIC MODEL FITTED TO POST-MODAL MORTALITY AMONG JAPANESE FEMALES, 1995-1999. (THE LEFT ASTERISK IS AT THE MODAL AGE AT DEATH AND THE PLUS MARK INDICATES THE POINT OF INFLECTION.)



Acknowledgement: We are grateful Sophie Le Roy for the data manipulation and technical support.