An Iteration Technique for Estimating Survival Probabilities at Ages Beyond 5 from a set of Cumulative Life Table Survival Ratios (T_{x+5} / T_x)

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Introduction:

The usual 5-year life table survival ratio (in short **5-LSR)** between the age-groups (x, x+5) and (x+5, x+10), which is defined by the ratio ${}_5\mathbf{L}_{x+5} / {}_5\mathbf{L}_x$ in life table terminology, is an important and useful statistics in demographic research and mortality analysis in particular. In countries having poor death registration statistics, conventionally the quantities **5-LSRs** are estimated indirectly from the population age-distribution of a country at two points of time 5 or 10 years apart, as it is generally believed that in many such countries the population age-data, collected through its population censuses or surveys, are relatively of better quality than the death registration statistics[1]. In such a conventional approach, as mentioned above, the quantities **5-LSRs** are estimated from the population age-distributions under the assumption of approximate equality[2] between the population survival ratios (**PSRs**), based on the quinquennial age-data at two enumerations 5 or 10 years apart, and the corresponding life table survival ratios (**LSRs**). It has been shown elsewhere by the author (Lahiri, 2004) that in a *closed* destable population, which is neither stationary nor stable, the 5-LSRs can be estimated indirectly from the age-distributions at two points of time at an interval not necessarily multiple of 5 through the use of generalized model of age-structure (Preston and Coale, 1982) applicable to any population[3] closed or open.

In an earlier paper of the author presented in the annual meeting of PAA 2004 (Lahiri, 2004), it was shown that 5-year life table ***cumulative*** survival ratios (in short **5-cum-LSRs**), defined by the ratio ($\mathbf{T}_{x+5} / \mathbf{T}_x$) in life table terminology, can be estimated from the cumulated age-

[1] The incomplete death registration statistics can be used for life table construction provided, of course, the relevant data are suitably adjusted for completeness (see Bennett and Horiuchi, 1981 & 1984; Preston and Lahiri, 1991; and Bhat, 2002).
[2] Such equality holds good only when the population under study is either stationary or stable (see Lahiri, 2004, and Lahiri and Menezes, 2004).
[3] Such a population model will be henceforth called Generalized Population Model (GPM).

data (population at ages 5+, 10+, 15+, etc.) at two enumerations, separated by any time-interval (not necessarily multiple of 5), through a formula derived under the ***generalized population model*** **(GPM)** applicable to any closed population. It is frequently found in many developing countries that such non-conventional survival ratios run quite smoothly over ages even in the presence of heavy age misreporting in censuses or surveys. Whereas, under such a situation the 5-year usual life table survival ratios (in short **5-LSRs**) estimated from the enumerated age-data at two points of time through the use of GPM, behave rather erratically and at times exceed unity which is rather absurd in a closed population. These usual survival ratios cannot, therefore, be used to construct adult mortality table without radical smoothing of the age-data or these survival ratios. Keeping in mind that the available smoothing procedures are rather arbitrary, subjective and often influenced by personal predilections, attempt was made by the author (Lahiri, 1983) in developing a technique for constructing adult mortality table which does not require radical smoothing of the age-data or the raw life table survival ratios – **5-LSRs** (see also Lahiri, 1985).

In the present paper an attempt has been made to develop a technique for estimating survival probability between ages x to x+5 (${}_5\mathbf{p}_\mathbf{x}$) from a set of non-conventional survival life table survival rations (**5-cum-LSRs**), as defined earlier. It may be noted that a given set of **5-LSRs**, denoted by ${}_5\mathbf{s}_{\mathbf{x+2.5}}\ \mathbf{(={}_5L_{x+5}/{}_5L_x)}$, determines a life table uniquely whereas a given set of **5-cum-LSRs** (${}_5\mathbf{s}_{\mathbf{x+}}$), defined by the ratio ${}_5\mathbf{s}_{\mathbf{x+}}\ \mathbf{=T_{x+5}/T_x}$, can give rise to numerous sets of ${}_5\mathbf{p}_\mathbf{x}$ values – not all satisfying the usual properties of ${}_5\mathbf{p}_\mathbf{x}$ -- column of a life table[4]. This is so because a given value of $\mathbf{T_x}$ can be obtained from various combinations of ${}_5\mathbf{L}_\mathbf{x}$ values beyond age <u>**x**</u> since $\mathbf{T_x} = \sum_{y=x}^{w} {}_5\mathbf{L}_\mathbf{y}$. The aim of this study is to identify an appropriate set of ${}_5\mathbf{p}_\mathbf{x}$-values which is not only consistent with the given set of **5-cum-LSRs** (${}_5\mathbf{s}_{\mathbf{x+}}$) but also satisfy the usual properties of life table survival probability and its pattern over ages. Interrelationships between life table functions and certain assumption of the nature of the $\mathbf{l_x}$ curve have been used for the purpose in view. The values of **5-cum-LSRs** (${}_5\mathbf{s}_{\mathbf{x+}}$) can be estimated directly from the enumerated age-data at

[4] The life table ${}_5\mathbf{p}_\mathbf{x}$ -- values, being survival probabilities, must lie between 0 and 1 for all ages and should follow a specific pattern. The ${}_5\mathbf{p}_\mathbf{x}$ value should increase initially as age increases up to certain age-interval, viz., 10-15 or 15-20 and thereafter it should decline as age increases.

two points of time (not necessarily 5 or 10 years apart) through the method proposed by the author earlier (see, Lahiri, 2004; and see also Lahiri and Menezes, 2004). One may make use of a preliminary adult mortality table obtained through the census-based method proposed by Preston and Bennett (1983) or that proposed by Lahiri and Menezes[5] (2004) for estimating **5-cum-LSRs**. Such ${}_5\mathbf{s}_{x+}$ values based on unadjusted census age-returns should be smoothed or graduated before use. Since the present paper deals with the problem of estimating an appropriate set of ${}_5\mathbf{p}_x$-values at adult ages from a given set of (error free) **5-cum-LSRs** (${}_5\mathbf{s}_{x+}$), the questions of estimating ${}_5\mathbf{s}_{x+}$ values directly from two census age-returns and its smoothing or graduation are not discussed here. The necessary details on these issues may be found elsewhere (see, Preston and Bennett, 1983; and Lahiri, 2004). The following sections deal with the development of various formulas relating ${}_5\mathbf{p}_x$'s for a given set of ${}_5\mathbf{s}_{x+}$ values.

Methodology:

Two approaches have been proposed in this paper for estimating ${}_5\mathbf{p}_x$ at various quinquennial ages from a given set of ${}_5\mathbf{s}_{x+}$ values at various quinquennial ages beyond some childhood age. In the ***first approach*** a sufficiently good approximation of ${}_5\mathbf{p}_x$ at age '**a**' say was obtained first from a given set of ${}_5\mathbf{s}_{x+}$ values under the assumption that $\mathbf{l}_x$ is a linear function of x in the 10-year age-interval (x-5, x+5). Later on this ${}_5\mathbf{p}_a$ value in conjunction with an algebraic chain relationship between ${}_5\mathbf{p}_x$'s in two adjacent age-intervals for a given set of ${}_5\mathbf{s}_{x+}$ values, derived under the assumption of linearity of $\mathbf{l}_x$ curve within each of the adjacent 5-year age-intervals, are used to obtain all the ${}_5\mathbf{p}_x$ under age '**a**' and above age '**a**'. In the ***second approach an iterative procedure*** has been proposed in estimating all the ${}_5\mathbf{p}_x$ under Greville's approximation of ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ values knowing a best estimate of ${}_5\mathbf{p}_{w-5}$, where '**w**' is the initial age of the terminal open-ended age-interval. It is worth mentioning here that the first approach is applicable under conventional approximation of ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ only.

[5] It may noted here that one can directly make use of the formula proposed by Lahiri (2004) for estimating **5-cum-LSRs** (${}_5\mathbf{s}_{x+} = \frac{\mathbf{T}_{x+5}}{\mathbf{T}_x}$) without constructing the adult mortality table as required in the case of Preston-Bennett (1983) method.

Among the variety of approximations for ${}_5\mathbf{L}_x$ in terms of $\mathbf{l}_x$, the following approximations are frequently used in life table costruction:

$$ {}_5\mathbf{L}_x = \tfrac{5}{2}(\mathbf{l}_x + \mathbf{l}_{x+5}) \text{ ....(1.0)} $$

and

$$ \left.\begin{array}{ll} {}_5\mathbf{L}_x & = \tfrac{65}{24}(\mathbf{l}_x + \mathbf{l}_{x+5}) - \tfrac{5}{24}(\mathbf{l}_{x-5} + \mathbf{l}_{x+10}) \\ \mathbf{or} & = \tfrac{5}{2}(\mathbf{l}_x + \mathbf{l}_{x+5}) + \tfrac{5}{24}({}_5\mathbf{d}_{x+5} - {}_5\mathbf{d}_{x-5}) \end{array}\right\} \text{ ....(1.1)} $$

The first formula (1.0) is popularly known as conventional approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$, obtained under the assumption of linearity of $\mathbf{l}_x$-curve. The second formula (1.1), which was proposed by Greville (1945) under the assumption of a cubic curve for $\mathbf{l}_x$ through the four points $\mathbf{l}_{x-5}$, $\mathbf{l}_x$, $\mathbf{l}_{x+5}$, and $\mathbf{l}_{x+10}$, is popularly known as Greville's approximation for ${}_5\mathbf{L}_x$.

<u>Approach-I</u>: Estimation of Survival Probabilities through an Algebraic Chain Relationship Between ${}_5\mathbf{p}_x$ and ${}_5\mathbf{p}_{x+5}$ Under the Conventional Approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$

Under the assumption of linearity of the $\mathbf{l}_x$-curve in the two successive 5-year age-intervals (x, x+5) and (x+5, x+10), one can find the following approximate chain relationships between ${}_5\mathbf{p}_x$ and ${}_5\mathbf{p}_{x+5}$ for a given set of ${}_5\mathbf{s}_{x+}$ values:

$$ {}_5\mathbf{p}_x \approx \frac{\left[1 - {}_5\mathbf{s}_{(x+5)+}\right] * {}_5\mathbf{s}_{x+}}{\left(1 + {}_5\mathbf{p}_{x+5}\right) * \left(1 - {}_5\mathbf{s}_{x+}\right) - \left[1 - {}_5\mathbf{s}_{(x+5)+}\right] * {}_5\mathbf{s}_{x+}} \text{ ....(2),} $$

and

$$ {}_5\mathbf{p}_{x+5} \approx \frac{{}_5\mathbf{s}_{x+} * \left[1 - {}_5\mathbf{s}_{(x+5)+}\right] * \left(1 + {}_5\mathbf{p}_x\right)}{\left(1 - {}_5\mathbf{s}_{x+}\right) * {}_5\mathbf{p}_x} - \mathbf{1} \text{ ....(3).} $$

The analytical justifications of the above approximations are given in the Appendix. Thus, if some how one can estimate a particular entity of the set of ${}_5\mathbf{p}_x$ values, namely, ${}_5\mathbf{p}_a$ (where the age **<u>a</u>** lies between ages 5 and **<u>w</u>**, '**<u>w</u>**' being the initial age of the open-ended terminal age-interval '**<u>w</u>** and above'), the values of ${}_5\mathbf{p}_x$ can be obtained through the repeated application of the formulas (2) and (3) for ages under **<u>a</u>** and above **<u>a</u>** respectively. Now, our problem is how to obtain a reasonably good estimate of ${}_5\mathbf{p}_a$, where $\mathbf{5 \le a \le w}$, from ${}_5\mathbf{s}_{x+}$ values alone. This has been discussed in the following section.

Estimation of ${}_5\mathbf{p}_x$ Values from a Known Set of ${}_5\mathbf{s}_{x+}$ Values Alone:

It is well known that a life table and hence ${}_5\mathbf{s}_{x+}$ values are fixed for a given set of ${}_5\mathbf{p}_x$ **(or** $\mathbf{l}_x$) values. But can we determine ${}_5\mathbf{p}_x$ **(or** $\mathbf{l}_x$) values uniquely from a given set of ${}_5\mathbf{s}_{x+}$ values alone? Apparently one may not find a positive answer to this problem, as a given set of ${}_5\mathbf{s}_{x+}$ values may give rise to numerous sets of ${}_5\mathbf{p}_x$ values, as mentioned earlier. An empirical investigation based on Coale-Demeny (1983) model life tables also support such findings (Lahiri, 1985). It has been pointed out earlier that the knowledge of $\mathbf{l}_x$ (or ${}_5\mathbf{p}_x$) values at all ages determine a life table uniquely. Thus, in addition to the knowledge of ${}_5\mathbf{s}_{x+}$ values at ages 5 and above, if we introduce certain condition on the nature of the $\mathbf{l}_x$ - curve which is frequently used in practice, one may find some solution to the problem mentioned above. Such an analytical investigation is shown below.

Under the assumption of linearity of $\mathbf{l}_y$ function in the 10-year age-interval (x-5, x+5) and (x, x+10), it can be shown that $\mathbf{l}_x$ and $\mathbf{l}_{x+5}$ can be approximated as follows[6]:

$$\mathbf{l}_x \approx \tfrac{1}{10}\left({}_5\mathbf{L}_{x-5} + {}_5\mathbf{L}_x\right) \;\ldots\ldots\ldots\ldots\ldots(3.1)$$

and

$$\mathbf{l}_{x+5} \approx \tfrac{1}{10}\left({}_5\mathbf{L}_x + {}_5\mathbf{L}_{x+5}\right)\ldots\ldots\ldots\ldots\ldots(3.2)$$

Now, using (3.1) and (3.2), one can easily find that ${}_5\mathbf{p}_x(=\mathbf{l}_{x+5}/\mathbf{l}_x)$ can be approximated through the following equation:

$$ {}_5\mathbf{p}_x \approx \frac{{}_5\mathbf{s}_{x-2.5} * \left[1 + {}_5\mathbf{s}_{x+2.5}\right]}{1 + {}_5\mathbf{s}_{x-2.5}}, \mathbf{10 \le x \le w - 5} \;\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where ${}_5\mathbf{s}_{x-2.5}\ (={}_5\mathbf{L}_x/{}_5\mathbf{L}_{x-5})$ and ${}_5\mathbf{s}_{x+2.5}\ (={}_5\mathbf{L}_{x+5}/{}_5\mathbf{L}_x)$ are the life table survival ratios between the 5-year age-intervals (x-5, x) to (x, x+5), and (x, x+5) to (x+5, x+10) respectively. Using the relationship between ${}_5\mathbf{L}_x$ and $\mathbf{T}_x$ columns of an abridge life table, that is, ${}_5\mathbf{L}_x = \mathbf{T}_x - \mathbf{T}_{x+5}$, one can

[6] There are of course various other forms of relationship between $\mathbf{l}_x$ and ${}_5\mathbf{L}_x$ depending upon the nature of $\mathbf{l}_x$ curve (for further discussions see, Arriaga, 1966 and Keyfitz, 1977).

easily show that **5-LSR**, that is ${}_5s_{x+2.5}$ $(={}_5L_{x+5}/{}_5L_x)$, can be expressed in terms **5-cum-LSR**, that is ${}_5s_{x+}$ $(=T_{x+5}/T_x)$ in two successive age-intervals through the following **exact** relationship:

$${}_5s_{x+2.5} = \frac{{}_5L_{x+5}}{{}_5L_x} = \frac{{}_5s_{x+} * \lfloor 1 - {}_5s_{(x+5)+} \rfloor}{1 - {}_5s_{x+}} \ldots\ldots\ldots\ldots(5),$$

for x = 5, 10, 15, ……., w-10, & w-5. The identity (5) shows that knowing the true values of ${}_5s_{x+}$ how one can obtain the true values of ${}_5s_{x+2.5}$ which determine a life table uniquely provided an initial value of ${}_5L_x$ (say at age x = a) is known in advance. Alternately, the true values of ${}_5s_{x+}$ can also be used directly to estimate ${}_5p_x$ values under certain assumption of linearity l_x- curve through the formulas (2) and (3), provided a particular value of ${}_5p_x$ is known in advance.

Now, using (5) in (4) we get the following approximation for ${}_5p_x$:

$${}_5p_x \approx \frac{{}_5s_{(x-5)+} * \lfloor 1 - {}_5s_{x+} * {}_5s_{(x+5)+} \rfloor}{1 - {}_5s_{(x-5)+} * {}_5s_{x+}}, \text{ for } 10 \le x \le w-5 \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Thus, the equation (6) provides a formula for estimating ${}_5p_x$ from ${}_5s_{x+}$ values alone under constraints (3.1) and (3.2). At this juncture, one may raise question regarding the utility of the equations (2) and (3) when ${}_5p_x$ values can easily be obtained from ${}_5s_{x+}$ values alone through the equation (6). In this context it may be noted that the development of the equation (6) requires more restrictive assumptions than those of the equations (2) and (3). More specifically, the formula (6) provides reasonably good estimates of ${}_5p_x$**'s** from the values of ${}_5s_{x+}$ as long as l_x is approximately a linear function of age **x** and the same linear function operates in the two partially overlapping 10-year age-intervals (x-5, x+5) and (x, x+10). On the other hand the formula (2) or (3), which provides a chain relationship between ${}_5p_x$ and ${}_5p_{x+5}$ for a given set of ${}_5s_{x+}$ values beyond age 5, is valid under the assumption of linearity of l_x - curve in two consecutive non-overlapping 5-year age-intervals[7] (x-5, x) and (x, x+5). This point will be examined later on the basis of an empirical experimentation. For estimating ${}_5p_x$ values at various quinquennial ages in the age-span -- ‘5 and above’ from a given set of ${}_5s_{x+}$ values beyond age 5, one may make use of formula (2) and (3),

[7] The same linear function should be applicable in the two adjacent but non-overlapping age-intervals (x-5, x) and (x, x+5) so as to obtain sufficiently precise estimate of ${}_5p_x$.

however, it requires the knowledge of a reasonably good estimate of ***one*** of the ${}_5\mathbf{p}_x$ values, say at age '<u>a</u>', that is ${}_5\mathbf{p}_a$, and for this purpose we need to use the formula (6). However, the accuracy of the estimated value of ${}_5\mathbf{p}_a$ depends upon the validity of the assumption of linearity of $\mathbf{l}_x$ - curve in the two partially overlapping 10-year age-interval (a-5, a+5) and (a, a+10). Thus, it would be worth examining empirically the best possible estimate of ${}_5\mathbf{p}_x$ for a particular age-interval in a life table constructed under the assumption of linearity of $\mathbf{l}_x$ within each of the two adjacent non-overlapping 5-year age-intervals but such an assumption of linearity may not necessarily be true in two partially overlapping 10-year age-intervals which is assumed while estimating ${}_5\mathbf{p}_x$ values from a given set of ${}_5\mathbf{s}_{x+}$ values alone through the formula (6). Such an empirical investigation based on Coale-Demeny (C-D) West model life tables, which have been constructed under the conventional assumption, shows that the error in estimating ${}_5\mathbf{p}_x$ in the age-range (15, 30) through the formula (6) is much lower compared to the other values of ${}_5\mathbf{p}_x$'s obtained through the application of the formula (6) for a given set of ${}_5\mathbf{s}_{x+}$ values beyond age 5. However, the estimate of ${}_5\mathbf{p}_{15}$, which is sufficiently close to true one, may be regarded as reasonably accurate for all practical purposes. The details of the empirical investigation with respect to C-D west life table system can be found in the next section[8]. Thus, knowing a reasonably good estimate of ${}_5\mathbf{p}_{15}$ the other values of ${}_5\mathbf{p}_x$ can be obtained by repeated application of the formula (2) for ages below 15 and the formula (3) for ages above 15 for a given set of ${}_5\mathbf{s}_{x+}$ values beyond age 5, and hence one can easily construct adult life table beyond age 5 through conventional approach as usual keeping in mind that the value of $\mathbf{T}_w$ (<u>**w**</u> being the initial age of the last open-ended age-interval) can be obtained through use of the following exact identity applicable between life table columns-- ${}_5\mathbf{L}_x$ and $\mathbf{T}_x$:

$$\mathbf{T}_x = \left[\frac{{}_5\mathbf{s}_{(x-5)+}}{1-{}_5\mathbf{s}_{(x-5)+}}\right] * {}_5\mathbf{L}_{x-5} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The above exact relationship (identity 7) can be obtained keeping in mind the fundamental relationship between ${}_5\mathbf{L}_x$ and $\mathbf{T}_x$ columns of a life table and ${}_5\mathbf{s}_{x+} = \mathbf{T}_{x+5} / \mathbf{T}_x$. One of the vexing

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[8] Similar investigations had also have also been carried out with respect to other C-D regional model life table systems, and the same result was found in the other regional model life tables excepting that of North model where the estimate of ${}_5\mathbf{p}_{20}$ provides the best result. The empirical investigations with respect to the other than C-D model life table systems are not presented here.

issue in life table construction is how to obtain $\mathbf{L}_{\mathbf{w+}}$(or $\mathbf{T}_{\mathbf{w}}$) for the open-ended terminal age-interval '<u>**w**</u> and above'. This problem[9] can be easily tackled in the present situation with knowledge of ${}_5\mathbf{s}_{(\mathbf{w-5})+}$ through the formula (7), provided of course we know the value of ${}_5\mathbf{L}_{\mathbf{w-5}}$.

It may be mentioned here that the procedure for estimating adult mortality table requires the smoothed series of ${}_5\mathbf{s}_{\mathbf{x+}}$ values. A great advantage of using 5-**cum-LSRs** (${}_5\mathbf{s}_{\mathbf{x+}}$) instead of **5-LSRs** (${}_5\mathbf{s}_{\mathbf{x+2.5}}$) for estimating ${}_5\mathbf{p}_{\mathbf{x}}$**'**s is that even when the values of ${}_5\mathbf{s}_{\mathbf{x+2.5}}$ are rather erratic due to age-misreporting or otherwise[10], the values of ${}_5\mathbf{s}_{\mathbf{x+}}$ follows a regular declining pattern as a result of dampening effect of cumulation of ${}_5\mathbf{L}_{\mathbf{x}}$ values over ages. Therefore, smoothening of the 'observed'[11] ${}_5\mathbf{s}_{\mathbf{x+}}$ values, which follow a specific pattern (consistent with the life table ratio $\mathbf{T}_{\mathbf{x+5}}\,/\,\mathbf{T}_{\mathbf{x}}$) over ages in contrast to the 'observed' ${}_5\mathbf{s}_{\mathbf{x+2.5}}$ values which are likely to be rather erratic in nature mostly due to age-misreporting, can be carried out more conveniently and scientifically than those highly erratic 'observed' ${}_5\mathbf{s}_{\mathbf{x+2.5}}$ values. Another interesting feature is that the 'observed' ${}_5\mathbf{s}_{\mathbf{x+}}$ values invariably lie between 0 and 1 for all ages as that of the life table ratio $\mathbf{T}_{\mathbf{x+5}}\,/\,\mathbf{T}_{\mathbf{x}}$ whereas the 'observed' ${}_5\mathbf{s}_{\mathbf{x+2.5}}$ values become so erratic at times that it may exceed unity which is rather absurd in a closed population. Thus, the 'observed' ${}_5\mathbf{s}_{\mathbf{x+}}$ values where logit transformation can be applied conveniently as the 'observed' ${}_5\mathbf{s}_{\mathbf{x+}}$ values lie between 0 and 1 for all ages can be smoothed or graduated through a Brass-type two-parameter logit model. Since the aim of this study is to develop a technique for estimating the appropriate set of ${}_5\mathbf{p}_{\mathbf{x}}$ values at ages beyond age 5 from a given set of ${}_5\mathbf{s}_{\mathbf{x+}}$ values beyond age 5, which are assumed to be sufficiently accurate, the graduation of the observed set of ${}_5\mathbf{s}_{\mathbf{x+}}$ values has not been discussed here.

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[9] Even in situation where the life table is based on the age-specific death rate, one may identify an approximate model life table (MLT) consistent with the '**p-column' (${}_5\mathbf{p}_{\mathbf{x}}$ values)** and this MLT may be used to get an approximate value of ${}_5\mathbf{s}_{(\mathbf{w-5})+}$ under the assumption that **5-cum-LSR** at some old age is unlikely to change drastically over time.

[10] Apart from the errors in age-reporting, the assumption involved in estimating life table survival ratios from the discrete age-data of a non-stationary population following generalized population model might also introduce some distortions in the smoothness of **5-LSRs.**

[11] The term 'observed' ${}_5\mathbf{s}_{\mathbf{x+}}$ values stand for those obtained from a preliminary life table based on two consecutive census age-returns of a population at ages 5+, 10+, 15+, etc., through the use of the formula proposed by Lahiri (2004) under the generalized model of age-structure.

Empirical Investigations Using Coale-Demeny (C-D) Model Life Tables for Estimating Survival Probabilities ($_5p_x$) Over Ages for a Given Set of 5-cum-LSRs ($_5s_{x+}$) Beyond Age 5 Through Formula (6) along with the Formulas (2) and (3):

To appreciate the role of the formula (6) along with the formulas (2) and (3) in estimating $_5\mathbf{p}_x$ values, an empirical investigation has been carried out on the basis of C-D (1983) West model life tables for females. The relevant results are presented in the Tables 1 and 2.

(Table-1 to be presented here)

The Table-1 presents the estimates of $_5\mathbf{p}_x$ values obtained directly from a given set of $_5\mathbf{s}_{x+}$ values through the use of the formula (6) along with its relative error. Examining critically the data presented in Table-1, it is found in general the formula (6) provides reasonably accurate estimates of $_5\mathbf{p}_x$'s in the age-range 15-30.More precisely the relative error in estimating $_5\mathbf{p}_x$ values in the age-range (15, 30) is less than 0.17 percent; and at some ages the relative errors are even less than 0.01 percent. Another interesting feature is that the preciseness of the estimates increases as the level of life expectancy increases. A close study of the relative percentage errors ($_5\mathbf{ER}_x$) further reveals that the value of $_5\mathbf{p}_{15}$ estimated through the formula (6) seems to be reasonably precise in terms of relative percentage error (less than 0.01%). Ranking[12] the $_5\mathbf{ER}_x$ column in Table-1 in ascending order, it is found that the rank of $_5\mathbf{ER}_{15}$ is either 1 or 2 in all the life tables excepting that at level 5 where the rank of $_5\mathbf{ER}_{15}$ is 3. This value of $_5\mathbf{p}_{15}$ can be used, in turn, to estimate other value $_5\mathbf{p}_x$ through the repeated application of the formula (2) for ages below 15, and the formula (3) for ages above 15. The summary results, based on the formulas (2) and (3), are shown in the Table-2.

(Table-2 to be here)

A close examination of the data presented in Table-2 which is self explanatory reveals that the minimum value and range of variation of the percentage error in estimating $_5\mathbf{p}_x$ values at ages 5

[12] It may be noted that the rank 1 (one) is assigned to the value of $_5\mathbf{ER}_x$ having the lowest magnitude.

and above corresponding to the age 15 are both either lowest or the 2nd lowest among the respective figures at other ages for a given mortality level. This indicates that the estimates of $_5\mathbf{p}_x$**'s** at ages '5 & above', which has been obtained through the repeated application of the formulas (2) and (3) started with the value of $_5\mathbf{p}_{15}$ estimated through the formula (6), may be considered sufficiently accurate. The final estimates of $_5\mathbf{p}_x$**'s** at ages '5 & above', obtained through the above procedure, along with the errors in estimating the values of $_5\mathbf{p}_x$ over various mortality levels are presented in Table 3. Knowing that the values of $_5\mathbf{p}_x$**'s** can be obtained directly from the set of $_5\mathbf{s}_{x+}$ values through the equation (6) alone, one would be curious to know the special merit of the latter procedure which makes use of the equations (2) and (3) in conjunction with the equation (6) in estimating $_5\mathbf{p}_x$ values from the set of $_5\mathbf{s}_{x+}$ values. Comparing the $_5\mathbf{ER}_x$ column in Table-1 to that of Table 3, one can easily find that the errors in estimating $_5\mathbf{p}_x$ values in the latter procedure are quite small and much less than those the former.

(Table 3 to be here)

Similar investigations, based on C-D West model life table system for females as that mentioned above, have been carried out with respect to other set of model life table systems (viz., East, North, and South) prepared by Coale and Demeny (1983) for both the sexes. The detailed investigations (not shown here) indicate that in contrast to the West model life tables, in general, it is not possible to identify a particular estimate of $_5\mathbf{p}_a$ (for example, $_5\mathbf{p}_{15}$ in the case of West model life table system) through the formula (6), which in turn provide most reasonable estimates of $_5\mathbf{p}_x$**'s** at ages 5 & above through the repeated application of the formulas (2) and (3). However, it is found that on an average set of estimates of $_5\mathbf{p}_x$**'s** (geometric mean) of the three sets of $_5\mathbf{p}_x$ values at ages '5 & above' which were obtained successively through the application of the formulas (2) and (3) corresponding to each of the estimated values of $_5\mathbf{p}_{10}$, $_5\mathbf{p}_{15}$ **&** $_5\mathbf{p}_{20}$ based on the formula (6) alone provides reasonably good estimates of $_5\mathbf{p}_x$**'s** at ages '5 & above' for a given set of $_5\mathbf{s}_{x+}$ values. In other words, mathematically,

$_5\hat{\mathbf{p}}_x \approx \left({}_5\mathbf{p}_x(\mathbf{10}) * {}_5\mathbf{p}_x(\mathbf{15}) * {}_5\mathbf{p}_x(\mathbf{20})\right)^{\frac{1}{3}}$ for x=5, 10, 15,-------,w-10 & w-5 ………..(8), where ${}_5\mathbf{p}_x\mathbf{(a)}$, $\mathbf{(x \neq a)}$ refers to the estimates of ${}_5\mathbf{p}_x$ obtained through the application of the formula (2) or (3) knowing the value of ${}_5\mathbf{p}_a$ through the application of the formula (6).

Approach-II: An Iteration Process for Estimating ${}_5\mathbf{p}_x$ values at Ages 5 and above for a Known set of ${}_5\mathbf{s}_{x+}$ values

One easily verify the existence of the following mathematical exact relationship (identity) among life functions on which the iteration process, proposed here, is based:

$${}_5\mathbf{p}_x = {}_5\mathbf{s}_{x+} / {}_5\mathbf{E}_x \ldots\ldots\ldots (9)$$

The statistics ${}_5\mathbf{s}_{x+}$ has already been defined earlier. The quantity ${}_5\mathbf{E}_x$ in the above equation is defined by the ratio $\mathbf{e}^0_{x+5} / \mathbf{e}^0_x$, where $\mathbf{e}^0_x$ denotes life expectancy at age **x** according to the standard life table terminology**.** *The statistic 1- ${}_5\mathbf{E}_x$ may be interpreted as a measure of relative change in (future) the overall mortality level of a cohort of persons at exact age **x** over the age-interval (x, x+5) under the assumption that the cohort (unaffected by migration) is exposed to a fixed age-schedule of mortality throughout the life span beyond age **x**.* The above exact relationship (9) holds true in any life table irrespective of whether it has been constructed through ***conventional method*** or ***any other non-conventional approaches***, such as, Greville's (1945) method, which does not assume the linearity of $\mathbf{l}_x$ - curve. Under the ***conventional method*** based on the assumption of linearity of linearity of $\mathbf{l}_x$ - curve, one can easily make use of the algebraic chain relationships (2) and (3) along with the formula (6) mentioned under the Approach-I for estimating ${}_5\mathbf{p}_x$ values at ages 5 and above from a given or known set of ${}_5\mathbf{s}_{x+}$ values and hence the adult life table. However under ***a non-conventional approach***, which does not assume the linearity of linearity of $\mathbf{l}_x$ - curve, the use of the iterative procedure, mentioned below, has considerable importance in estimating the ${}_5\mathbf{p}_x$ values at ages 5 and above from a given (or known) set of ${}_5\mathbf{s}_{x+}$ values. ***The importance of the iterative procedure under the non-conventional approaches will be discussed later on.***

The iterative procedure makes use of the identity (9) with a known or observed set of ${}_5\mathbf{s}_{x+}$ values together with an initial approximation to the set of ${}_5\mathbf{E}_x$ values over ages. In the subsequent discussions the '${}_5\mathbf{p}_x$ values' and '${}_5\mathbf{E}_x$ values' will be simply called **'p-values'** and **'E-**

values' respectively. Corresponding to an initial approximation to the set of **'E-values'** over ages[13], the first approximated set of **'p-values'** over ages can be obtained through the equation (9) for a known or observed set of ${}_5\mathbf{s}_{x+}$ values over ages. Such a observed set of ${}_5\mathbf{s}_{x+}$ values over ages will be hence denoted by $\{{}_5\hat{s}_{x+}\}$. A revised set of **'E-values'** over ages can be computed from the $\mathbf{e}_x^0$- column of the first approximated life table [**LT(1)**] constructed on the basis of the first approximated set **'p-values'** over ages. This revised set of **'E-values'** together with the original observed set $\{{}_5\hat{s}_{x+}\}$ when used in the basic equation (9) produces the second approximated set of **'p-values'** over ages which, in turn, generates the second approximated life table – **LT(2)**. This life table **LT(2)** will yield the second revised set of **'E-values'** which can be used to obtain another new set of **'p-values'** through the equation (9) and hence the corresponding approximated life table. The iterative procedure based on the identity (9) can be expressed through the following equation:

$$ {}_5\mathbf{p}_x^{(i)} = {}_5\hat{\mathbf{s}}_{x+} / {}_5\mathbf{E}_x^{(i-1)} \ldots\ldots\ldots\ldots\ldots(9.1), $$

where the superscript '**i**' and '**i**-1' attached to ${}_5\mathbf{p}_x$ and ${}_5\mathbf{E}_x$ stand for the corresponding values at the ith and (i-1)th iterations. It may be noted that ${}_5\mathbf{E}_x^{(0)}$ stands for the initial value of ${}_5\mathbf{E}_x$. It must be emphasized here that for obtaining the **'p-values'** at the ith iteration the original observed set of ${}_5\mathbf{s}_{x+}$ values, that is $\{{}_5\hat{s}_{x+}\}$, need to be divided by the corresponding set of **'E-values'** at the (i-1)th iteration.

The above procedure may be repeated in the following cyclical order, -- **'p-values', LT, 'E-values'** – as many times as required until the two consecutive approximations to the set of **'E-values'** at each and every age became sufficiently close to each other. Let us consider that the observed values ${}_5\hat{s}_{x+}$ are available at ages **a, a+5, a+10,…….,w-5, and w,** where **a** being some childhood other 0 and 1, and '**w**' being the initial of the open-ended terminal age-interval. Mathematically, the iteration process will be terminated when the following condition holds good:

$$ \left\| \frac{{}_5\mathbf{E}_x^{(i-1)} - {}_5\mathbf{E}_x^{(i)}}{{}_5\mathbf{E}_x^{(i-1)}} \right\| \le \boldsymbol{\varepsilon} \text{ for x = a, a+5,……..,w-5, and w -------- (10)} $$

[13] A procedure for obtaining the initial approximation to the set of **'E-values'** over ages will be discussed later.

where ${}_5\mathbf{E}_x^{(i-1)}$ and ${}_5\mathbf{E}_x^{(i)}$ denote the values of ${}_5\mathbf{E}_x$ derived from the (i-1)th and ith approximated life tables respectively and $\boldsymbol{\varepsilon}$ is a pre-assigned positive fraction, however small. The operational procedure of the iteration process can explained through the **flowchart** shown under diagram-1.

(Flowchart to be here)

In addition to the realization of the above inequality (10), if the two consecutive sets of ${}_5\mathbf{s}_{x+}$ values calculated from the (i-1)th and ith approximations - $\left\{{}_5\mathbf{s}_{x+}^{(i-1)}\right\}$ and $\left\{{}_5\mathbf{s}_{x+}^{(i)}\right\}$ corresponding to the sets $\left\{{}_5\mathbf{E}_x^{(i-1)}\right\}$ and $\left\{{}_5\mathbf{E}_x^{(i)}\right\}$ respectively become also almost identical to the each other and the same time they are also identical to the original observed set $\left\{{}_5\hat{\mathbf{s}}_{x+}\right\}$, then one would have a fair amount of confidence in the reliability of the process.

In this study, the iteration process has been carried out with respect to two methods of life table construction – (i) **conventional method** that assumes a ***linearity*** of $\mathbf{l}_x$ - curve, (ii) Greville's method, which assumes a ***non-linearity (cubic curve)*** of $\mathbf{l}_x$ - curve. Though there is no specific advantage of the iteration process under conventional method, however, it has been carried here so as to ensure empirically that both the procedures (described under Approaches-I & II) leads to the same result. It is worthwhile to mention here that the ***iteration process under Greville's method*** has a special advantage in estimating ${}_5\mathbf{p}_x$ values from a set of a given or known set of ${}_5\mathbf{s}_{x+}$ values over that under the **conventional method.** This is primarily because there is no simple algebraic chain relationship between two successive ${}_5\mathbf{p}_x$**'s** for a given set of ${}_5\mathbf{s}_{x+}$ values under the ***Greville's method*** compared to that under the ***conventional method***[14]. Empirical examinations of the iteration process, discussed later on, under the conventional and Greville's methods will provide more insight in this matter. Each of the iterated life tables has been constructed by taking $\mathbf{l}_a = 100{,}000$ as the radix along with the corresponding iterated set of ${}_5\mathbf{p}_x$ values. Irrespective of the method life

[14] Under the Greville's method, which follows a cubic curve for $\mathbf{l}_x$, one can develop a more complicated algebraic relationship among four successive ${}_5\mathbf{p}_x$**'s** in contrast to a relatively simple algebraic relationship among two successive ${}_5\mathbf{p}_x$**'s** for a given set of ${}_5\mathbf{s}_{x+}$ values under conventional method which assumes linearity of $\mathbf{l}_x$ -curve (see formula 12).

table construction (conventional or Greville's method) beyond some childhood age '**a**' **(≥4),** the $\mathbf{T_x}$ - column of a life table at any iteration can be obtained as usual after estimating the last entity of $\mathbf{T_x}$ - column at i^th^ iteration, that is, $\mathbf{T_w^{(i)}}$ through the identity (7) as expressed by the following equation:

$$\mathbf{T_w^{(i)}} = \frac{{}_5\hat{\mathbf{s}}_{(w-5)+}}{1-{}_5\hat{\mathbf{s}}_{(w-5)+}} * {}_5\mathbf{L}_{w-5}^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where the superscript (i) attached to the life table functions denotes the value of the corresponding LT functions at the i^{th} iteration. It may be emphasized here while obtaining $\mathbf{T_w^{(i)}}$ from $\mathbf{L_{w-5}^{(i)}}$ at the i^{th} iteration we make use of ${}_5\hat{\mathbf{s}}_{(w-5)+}$ (the original observed value of ${}_5\mathbf{s}_{x+}$ at age w-5) but not ${}_5\mathbf{s}_{(w-5)+}$ corresponding to the i^{th} iterated LT.

Empirical Investigation and Importance of the Iterative Procedure

For better comprehension of the iterative procedure for construction of an abridged life table beyond age '**a**' (the age '**a**' is taken as 5 and 4) from an observed set of ${}_5\mathbf{s}_{x+}$ values, it is necessary to examine the iterative procedure empirically through a known life table. For the purpose of an empirical investigation of the procedure, a life table at age 5 onwards corresponding to Coale-Demeny (C-D) ***West*** Model Life Table for **male** at level 14 is considered here. Henceforth, this life table at ages 5 and above will be called **W.M.L-14**. This test life table, which is based on ${}_5\mathbf{p}_x$ values at ages 5, 10, ………,60 and 65, has been selected arbitrarily from the C-D West model life tables just to examine the validity of the iteration process empirically. This does not mean that the iteration process is applicable only to the said model life table. It may be noted in this context that once the values of ${}_5\mathbf{p}_x$ are known, the life table is uniquely determined irrespective of whether the values of ${}_5\mathbf{p}_x$ are obtained from a model life table or any other life table based on actual mortality experience of a country provided, of course, the formula for obtaining ${}_5\mathbf{L}_x$ values remains the same in all the cases.

The values of ${}_5\mathbf{p}_x$, ${}_5\mathbf{s}_{x+}$ **and** ${}_5\mathbf{E}_x$ corresponding to this test life table are presented in Table-4. The ${}_5\mathbf{L}_x$ column of this life table has been obtained by using the conventional approximation, that is, ${}_5\mathbf{L}_x \approx \frac{5}{2}(\mathbf{l}_x + \mathbf{l}_{x+5})$ for the ages 5, 10, 15, ……, 60 and 65 taking $\mathbf{l}_5$=100,000. The value of

L_{70+}, that is, T_{70} has been obtained through the identity (7). Now, from this known life table one can calculate the true values of ${}_5s_{x+}$ as the ratio T_{x+5}/T_x, for x = 5, 10, 15, ……….,60 and 65. This set of ${}_5s_{x+}$ values, obtained from W.M.L-14, may be considered as the observed set $\{{}_5\hat{s}_{x+}$, x = 5, 10, 15,……….,60 and 65$\}$ for the purpose of examining the validity of the iteration process.

(Table-4 to be here)

The following observations are worth noting during the iteration process on the basis of various initial sets of ${}_5E_x$ values. Let us take ε equal to 10^{-6}, so as to decide whether the process will be continued or not, as indicated by the stopping rule[15] described by the inequality (10).

Empirical Observations on the Iteration Process under Conventional Approximations for ${}_5L_x$ from l_x

(i) If the ${}_5E_x$ values at all ages are also borrowed as the initial sets of ${}_5E_x$ values from the **W.M.L-14** life table which provides the observed set of ${}_5s_{x+}$ values, the process terminates almost immediately and it converges to the true life table (see Table 4). In other words, the final iterated values of ${}_5p_x$, ${}_5s_{x+}$ and ${}_5E_x$ are exactly identical to those of **W.M.L-14** (see Table 4 for various stages of iteration). This is quite expected because both the set of ${}_5\hat{s}_{x+}$ and ${}_5E_x^{(0)}$ values were taken from the same life table (i.e. **W.M.L-14**) and therefore, the set of ratios ${}_5\hat{s}_{x+}/{}_5E_x^{(0)}$ for x = 5, 10, 15,…..,60 and 65 will be exactly identical to the set of ${}_5p_x$ values corresponding to **W.M.L-14**. This is because of exact mathematical relationship between the three variables -- ${}_5p_x$, ${}_5s_{x+}$, and ${}_5E_x$ given by the equation (9).

(ii) If an initial set of ${}_5E_x$ values, which is entirely different for all ages from that of **W.M.L-14**, is chosen arbitrarily to start the iteration process it is found that the process terminates at certain stage of iteration; however, it does not converge to the true life table (see, Table 5). A very interesting feature is that the final iterated set of ${}_5s_{x+}$ values for x = 5, 10, 15, ……., 60, and 65 become

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[15] The main reason for choosing ε equal to 10^{-6}, though arbitrary, is to obtain the estimates ${}_5p_x$, ${}_5s_{x+}$, and ${}_5E_x$ correct up to the fifth places of decimal.

identical to that of **W.M.L-14**, though final iterated set of ${}_5\mathbf{p}_x$ values is entirely different from that of **W.M.L.-14**. Another interesting feature is that the set of values ${}_5\mathbf{s}_{x+2.5}\,(={}_5\mathbf{L}_{x+5}/{}_5\mathbf{L}_x)$ for x = 5,10,15,…….., 55 and 60 obtained from the final iterated life table is different than that of the known life table (that is, **W.M.L.-14**) even though the final iterated ${}_5\mathbf{s}_{x+}$ values are same as those of the original life table.

(Table-5 to be here)

(iii) If the initial set of ${}_5\mathbf{E}_x$ values is such that the last entity of the initial set, that is, ${}_5\mathbf{E}_{65}$ is exactly identical to that of the known life table (that is, **W.M.L.-14**) and the remaining values of ${}_5\mathbf{E}_x$'s in the initial set are chosen arbitrarily, it is found that the process terminates at certain stage of iterations and the most interesting fact is that it converges to the true life table (see table 6).

(Table-6 to be here)

Furthermore, in addition to ${}_5\mathbf{E}_{65}$, the last entity of the initial set of ${}_5\mathbf{E}_x$ values, which is exactly same as that of the known life table, if the other values of ${}_5\mathbf{E}_x$ for x = 5, 10, 15,……..,55 and 60 are fairly close to those of the known life table (**W.M.L.-14**), the iteration process converges more quickly to the true set true life table than other forms of ${}_5\mathbf{E}_x$ values chosen arbitrarily (not presented here).

(iv) If the initial set of ${}_5\mathbf{E}_x$ values is such that the magnitude of the last entity[16], that is ${}_5\mathbf{E}_{65}$ is not exactly same as that of the known life table (that is, **W.M.L.-14**), but very close to that known life table and the remaining values of ${}_5\mathbf{E}_x$'s of the initial set may have any arbitrary values, the process terminates at certain stage of iteration and the final iterated life table becomes very close to the true one, but not exactly identical to true (original) life table(not presented here).

The above observations are based on empirical investigations of the iteration process, which has been carried out under the conventional approximation (1.0) for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ at each stage of

[16] In this case the magnitude of ${}_5\mathbf{E}_{65}$ is taken as the average of those to **W.M.L-14 and W.M.L-15**.

iteration. One would raise question here when there exists an algebraic chain relationship[17], as shown under formulas (2), and (3), between two successive ${}_5\mathbf{p}_x$ values for a given set of ${}_5\mathbf{s}_{x+}$ values under the conventional approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ what is the utility of the iteration process under the conventional approximation? It has been mentioned earlier that such a simple chain relationship between two successive ${}_5\mathbf{p}_x$ values for a given set of ${}_5\mathbf{s}_{x+}$ values does not exist under the Greville's approximation (1.1) for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$. Thus one would have more confidence on the iteration process if we could show empirically the validity the process under the Greville's approximation also.

For the empirical validation of the iteration process under the Greville's approximation it is necessary to have a known abridged life table, which makes use of the Greville's approximations (1.1) for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$. True sets of ${}_5\mathbf{p}_x$, ${}_5\mathbf{s}_{x+}$, and ${}_5\mathbf{E}_x$ values corresponding to such an abridged life table are presented in Table (7), which has been constructed on the basis of the $\mathbf{l}_x$ values at quinquennial ages between 4 and 69 under the Greville's approximations (1.1) for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$. The $\mathbf{l}_x$ values at ages 9, 14,……..,64 and 69 have been interpolated from the known $\mathbf{l}_x$ values at the conventional quinquennial ages between 5 and 70 corresponding to the C-D West model life table at level 12 for males, constructed under the conventional approximation by using the ordinary six-term interpolation formula proposed by Beers (1944 & 1945). The value of $\mathbf{l}_4$ has been borrowed from the Coale-Demeny Regional Model life table (Coale and Demeny, 1966, Table XV, pp.42-43). The abridged life table, which is based on the above $\mathbf{l}_x$ values at various quinquennial ages between 4 and 69 estimated under Greville's approximation, will be denoted, henceforth, by W.M.L.-12 (GREV). From the Table- 7 one can obtain the values of ${}_5\mathbf{s}_{x+}$ for ages x = 4, 9,……………,59 and 64 which can be considered as the observed set **$\{{}_5\hat{\mathbf{s}}_{x+}$; x = 4, 9, 14,………,59 & 64$\}$** for the purpose of the empirical investigation.

(Table-7 to be here)

[17] However, to begin with the chain relationship we need to know a catalyst value of ${}_5\mathbf{p}_a$ through the approximation (6) from ${}_5\mathbf{s}_{x+}$ values alone.

Similar observations under the conventional approximation, as mentioned earlier, are also found while examining the iteration process on the basis of various initial sets of ${}_5\mathbf{E}_x$ values applied on the observed set **$\{{}_5\hat{\mathbf{s}}_{x+}$; x = 4, 9, 14,………,59 & 64}** with respect to the known life table under Greville's approximation, called here W.M.L.-12 (Grev.). The results are shown in the tables 7, 8, and 9. The noticeable difference between the two iteration processes based on the two observed sets, viz., **$\{{}_5\hat{\mathbf{s}}_{x+}$; x = 5, 10, 15,………,60 & 65} and $\{{}_5\hat{\mathbf{s}}_{x+}$; x = 4, 9, 14,………,59 & 64}** corresponding to the known life tables W.M.L.-14 and W.M.L-12 respectively is that the latter iteration process takes less number of iterations to converge as compared to the former one. One would expect if the initial set of ${}_5\mathbf{E}_x$ values is borrowed from the concerned known life table, constructed under a specific approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ (conventional or Greville's), from which the observed set of ${}_5\hat{\mathbf{s}}_{x+}$ values are obtained, the iteration process should converge to the original known life table at the very first iteration provided, of course, the process is carried out under the same approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ at each iteration. This is actually realized in the latter case (see Table 7) where the iteration process has been carried out under Greville's approximation corresponding to the life table W.M.L-12 (GREV). However, it is worth noting here that the faster convergence of the latter iteration procedure under the Greville's approximation does not necessarily prove that the Greville's formula for estimating ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ provides most accurate estimate[18] of ${}_5\mathbf{L}_x$, since it may happen that the extent of errors in obtaining $\mathbf{T}_x$ and $\mathbf{e}^0_x$ values are such that the ultimate error in the ratio ${}_5\hat{\mathbf{s}}_{x+}/{}_5\mathbf{E}_x$ becomes either negligibly small or cancel out totally during forming the ratios ${}_5\mathbf{s}_{x+}(=\mathbf{T}_{x+5}/\mathbf{T}_x)$, ${}_5\mathbf{E}_x(=\mathbf{e}^0_{x+5}/\mathbf{e}^0_x)$ and ${}_5\mathbf{p}_x(={}_5\hat{\mathbf{s}}_{x+}/{}_5\mathbf{E}_x)$. However, it is interesting to note that the iteration process stabilizes much faster when the process operates under the Grevilles approximation compared to that under the conventional approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$.

An interesting property of the iterative procedure is that for any arbitrary initial set of ${}_5\mathbf{E}_x$ values, the last entity of the two sets $\{{}_5\mathbf{E}_x\}$ and $\{{}_5\mathbf{s}_{x+}\}$, denoted by ${}_5\mathbf{E}_w$ and ${}_5\mathbf{s}_{w+}$, and hence

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[18] However, the Greville's formula undoubtedly provides better result than the conventional formula for obtaining ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$.

${}_5p_w\,(={}_5s_{w+}/{}_5E_w)$, the last entity of the set of ${}_5p_x$ values remain unaltered over iterations[19]. However, the other values of the set of ${}_5E_x$ values and hence the other values of the two sets of ${}_5s_{x+}$ and ${}_5p_x$ values continue to get modified over iterations until they converge or practically stabilize to fixed values from higher ages to lower ages sequentially. It is interesting to observe that such stabilization of the values of ${}_5E_x$, ${}_5s_{x+}$ and ${}_5p_x$ occurs from one older age group to the next lower sequentially starting from the age-group 65-69 when the iteration processes are carried out under the conventional approximation. Whereas when the iteration process is carried out under the Greville's approximation, such stabilization also occurs sequentially from higher to lower ages ***taking four consecutive age-groups together***, and thus the convergence or stabilization takes place much faster in the case of the latter compared to the former. This is because under the conventional approximation there exists an algebraic chain relationship between two successive ${}_5p_x$ values for a given set of ${}_5s_{x+}$ values as defined by the formula (2) or (3). Therefore, if somehow the last entity of the set of ${}_5p_x$ values get fixed the other values of ${}_5p_x$'s at lower ages get sequentially fixed through the repeated application of the formula (2). In other words, the formula (2) works inherently in the iteration process under conventional approximation. Whereas under Greville's approximation there is no such simple chain relationship exists as that of the formula (2) or (3) under conventional approximation, however, it can be shown mathematically that there exists a relatively complicated algebraic relationship of the following form among four successive ${}_5p_x$ values for a given set of ${}_5s_{x+}$ values:

$$ {}_5p_x \approx \frac{A({}_5p_{x+5},\, {}_5p_{x+10},\, {}_5p_{x+15}) * {}_5s_{(x+5)+}}{1 - B({}_5p_{x+5},\, {}_5p_{x+10},\, {}_5p_{x+15}) * {}_5s_{(x+5)+}} \;\ldots\ldots\ldots\ldots\ldots\; (12), $$

where A and B are some coefficients which are independent of ${}_5p_x$ but they depend on ${}_5p_{x+5}$, ${}_5p_{x+10}$ and ${}_5p_{x+15}$ for a know value of ${}_5s_{(x+10)+}$. (A mathematical justification of such an approximation will be provided on request). Thus, the above relationship (12) among four consecutive ${}_5p_x$ values works inherently in the iteration process under Greville's approximation. It should be kept in mind that the iteration process is such that the last entity of the set of ${}_5E_x$ values

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[19] It can be shown mathematically that the invariant properties of the last entity of the initial set of ${}_5E_x$ values and those of the last two entities of the set of ${}_5s_{x+}$ and ${}_5p_x$ values over iteration are governed by the relationship (11).

and hence the last entity of the set of ${}_5\mathbf{p}_x\ (={}_5\hat{\mathbf{s}}_{x+}/{}_5\mathbf{E}_x)$ values get fixed over iterations due the exact mathematical relationship (11). And once the last four entities of the set of ${}_5\mathbf{p}_x$ values get fixed over the iteration process controlled by the relations (11) and (12), the other lower values of ${}_5\mathbf{p}_x$**'s** get fixed much faster over iterations under the Greville's approximation compared to those under the conventional approximation.

The Importance of the Iteration Process under the Greville's Approximation for Estimating the ${}_5\mathbf{p}_x$ Values from a Given Set of ${}_5\mathbf{s}_{x+}$ values

The above discussions clearly indicate that there is no special advantage of the use of the iteration process under the conventional approximation for estimating ${}_5\mathbf{p}_x$ values for a given set of ${}_5\mathbf{s}_{x+}$ values. This is because once we know the value any one entity of the set of ${}_5\mathbf{p}_x$ values through the formula (6) the other values of the set can be obtained easily through the repeated application of the chain relationships (2) and/or (3) for a given (or known) set of ${}_5\mathbf{s}_{x+}$ values. On the other hand with the knowledge of a particular value of ${}_5\mathbf{E}_x$ or ${}_5\mathbf{p}_x$**,** the algebraic chain relationship (12) alone for a given set of ${}_5\mathbf{s}_{x+}$ values cannot be used directly to estimate the other ${}_5\mathbf{p}_x$ values without the help of the iteration process. It is worthwhile to mention here if some how the values of ${}_5\mathbf{p}_{65}$, ${}_5\mathbf{p}_{60}$ **and** ${}_5\mathbf{p}_{55}$ get estimated or fixed in advance, the remaining values of ${}_5\mathbf{p}_x$, that is, ${}_5\mathbf{p}_{50}$**,** ${}_5\mathbf{p}_{45}$, ${}_5\mathbf{p}_{55}$ **, ……,** ${}_5\mathbf{p}_{10}$ **and** ${}_5\mathbf{p}_5$ can be obtained successively by the repeated application of the formula (12) under the Greville's approximation. Furthermore, it may be noted the formula (12) is much more complicated to use in practice even if the last three consecutive ${}_5\mathbf{p}_x$ values are known than the formula (2) or (3) derived under conventional approximation for ${}_5\mathbf{L}_x$. On the other hand, surprisingly the iteration process under Greville's approximation for ${}_5\mathbf{L}_x$ works beautifully even if only one value of ${}_5\mathbf{E}_x$, namely ${}_5\mathbf{E}_{64}$, the last value of the set $\{{}_5\mathbf{E}_x;\mathbf{x = 4, 9, 14,......,59, 64}\}$ along with known set of ${}_5\mathbf{s}_{x+}$ values are available irrespective of the magnitude of other ${}_5\mathbf{E}_x$**'s**. It appears that the iteration process under Greville's approximation for ${}_5\mathbf{L}_x$ is such that the last three values of ${}_5\mathbf{p}_x$**'s** get fixed over iteration through the method of successive approximations under the property of the iteration process that the last entity of the set of ${}_5\mathbf{p}_x$ values remains unaltered over iterations which is governed by the formula (11) for obtaining the ***last value*** of the column $\mathbf{T_x}$ at each iteration. It is worthwhile to mention here that the iteration process, described under the Approach-II, can be regarded as a one-parameter model, particularly under the Greville's approximation as the process depends only on the last entity of the initial set $\{{}_5\mathbf{E}_x^{(0)}\}$ which remains unaltered over iterations.

Estimation of the Last Entity of the Initial Set $\{{}_5\mathbf{E}_x^{(0)}\}$

Let us consider the requisite data are presented in the following quinquennial ages – a, a+5, a+10,………,w-5; where 'w' being the initial age of the open-ended terminal age-interval. The foregoing discussions on the basis of the empirical investigation clearly reveals that the iteration process converges exactly or closely to the true set of ${}_5\mathbf{p}_x$ values for a given (observed) set of ${}_5\mathbf{s}_{x+}$ values, only when the last entity of the initial set of ${}_5\mathbf{E}_x$ values, that is ${}_5\mathbf{E}_{w-5}^{(0)}$, becomes exactly or almost identical to that of the true set $\{{}_5\mathbf{E}_x^{(0)}\}$ even if the other values of the initial set of ${}_5\mathbf{E}_x$ values are chosen arbitrarily. In other words, only the true knowledge of the last entity (${}_5\mathbf{E}_{w-5}^{(0)}$)of the initial set $\{{}_5\mathbf{E}_x^{(0)}\}$ of the iteration process, described by the flowchart under diagram-1, sets the tune in determining the true life table and thereby the true set of ${}_5\mathbf{p}_x$ values uniquely consistent with a given (observed) set of ${}_5\mathbf{s}_{x+}$ values. Now our problem is how to obtain the, the ${}_5\mathbf{E}_{w-5}^{(0)}$, the last entity of the initial set of ${}_5\mathbf{E}_x$ values of the iteration process. There are various ways to obtain a reasonably good estimate of ${}_5\mathbf{E}_{w-5}^{(0)}$. In this paper a simple procedure has been presented as described below.

It has been shown earlier under the Approach-I while introducing an algebraic chain relationship between ${}_5\mathbf{p}_x$**'s** in two successive age-intervals, shown under the formulas (2) and (3), for the purpose of estimating the probability of survival (${}_5\mathbf{p}_x$) at various age-intervals we need to know a sufficiently reliable estimate of ${}_5\mathbf{p}_x$ at a particular age through the formula (6). It appears from the analysis (shown under Table-1) based on the formula (6) that in general the error in estimating ${}_5\mathbf{p}_x$ from ${}_5\mathbf{s}_{x+}$ values alone is much lower in the age range-range (15, 30) compared to those of the other age groups. However, on an average it seems that the estimation error is lowest for the age group 15-19. Thus we propose to have three sets of estimates of ${}_5\mathbf{p}^{(0)}_{w-5}$ through the repeated application of the formula (3) corresponding to each of the survival probabilities -- ${}_5\mathbf{p}_{15}$, ${}_5\mathbf{p}_{20}$ **and** ${}_5\mathbf{p}_{25}$ estimated through the equation (6). The geometric mean of these three sets of estimates of ${}_5\mathbf{p}^{(0)}_{w-5}$ may be taken as the final estimate of the last entity of the initial set of true set of ${}_5\mathbf{p}_x$ values. Mathematically, one may represent the final estimate of the initial set of true set of ${}_5\mathbf{p}_x$ values through the following approximation:

$$ {}_5\hat{\mathbf{p}}^{(0)}_{w-5} \approx \left({}_5\mathbf{p}^{(0)}_{w-5}(15) * {}_5\mathbf{p}^{(0)}_{w-5}(20) * {}_5\mathbf{p}^{(0)}_{w-5}(25) \right)^{1/3} \ldots\ldots\ldots\ (13) $$

where, ${}_5\mathbf{p}^{(0)}_{w-5}(15)$, ${}_5\mathbf{p}^{(0)}_{w-5}(20)$ **and** ${}_5\mathbf{p}^{(0)}_{w-5}(25)$ denote the estimates of ${}_5\mathbf{p}^{(0)}_{w-5}$, obtained through the repeated application of the chain relationship (3) starting with the values of ${}_5\mathbf{p}_{15}$, ${}_5\mathbf{p}_{20}$ **and** ${}_5\mathbf{p}_{25}$ estimated through the equation (6) respectively for a given set of ${}_5\mathbf{s}_{x+}$ values alone. Now, using this value of ${}_5\mathbf{p}^{(0)}_{w-5}$ as estimated through the approximation (13), one can easily get the estimate of the last entity of the true set of ${}_5\mathbf{E}_x$ values through the following formula, which follows from the identity (9).

$$ {}_5\hat{\mathbf{E}}^{(0)}_{w-5} = {}_5\hat{\mathbf{s}}_{w-5+} / {}_5\hat{\mathbf{p}}_{w-5} \ldots\ldots\ldots\ldots\ldots (14). $$

It may be noted the magnitudes of ${}_5\hat{\mathbf{E}}^{(0)}_{w-5}$, **and** ${}_5\hat{\mathbf{p}}^{(0)}_{w-5}$ remain unaltered over iterations. However, the other values of ${}_5\mathbf{E}_x$ **and** ${}_5\mathbf{p}_x$ get modified and stabilize sequentially from higher ages to lower ages over the iteration process as described by flowchart shown under the diagram-1.

Summary and Conclusion:

It has been shown in an earlier paper of the author (presented in the PAA 2004 annual meeting) for countries with poor vital registration statistics how the cumulation of enumerated census (or survey) age-returns at two points of time, which are considerably affected by age-misreporting, can be used for estimating adult mortality through the use of **5-cum-LSRs** $({}_5\mathbf{s}_{x+} = \mathbf{T}_{x+5} / \mathbf{T}_x)$ estimated from the cumulated age-data at ages 5+, 10+, 15+, etc., at two points of time (see, Lahiri, 2004). Generalized population model age-structure, proposed by Preston and Coale (1982), was used for the purpose in view along with the assumption that the age-specific growth curve followed a second-degree polynomial.

In this paper an attempt has been made in estimating survival probabilities $({}_5\mathbf{p}_x)$ at various quinquennial ages starting from age 5 from a given set of set of ${}_5\mathbf{s}_{x+}$ values. It is well known that conventional 5-year life table survival ratios, that is, **5-LSRs** $({}_5\mathbf{L}_{x+5} / {}_5\mathbf{L}_x)$ determine a life table uniquely whereas **5-cum-LSRs** may give rise to numerous sets of $_5\mathbf{p}_\mathbf{x}$**'s** (not all satisfying the usual properties of **p-column or p-values** of a life table). The aim of this study is to identify an appropriate set of **p-values** which is not only consistent with a given (or known) set of **5-cum-LSRs** but also satisfy the usual property and pattern of a life table **p-value** over ages under various assumptions of the nature of $\mathbf{l}_\mathbf{x}$-curve. Two approaches have been proposed in this paper for estimating ***survival probabilities*** at various quinquennial ages based on a given set of **5-cum-LSRs** values at various quinquennial ages. First one makes use of a ***chain relationship between two survival probabilities in two adjacent age-intervals*** for a given set of **5-cum-LSRs** under the assumption of linearity of $\mathbf{l}_\mathbf{x}$-curve**,** and the other one is based on an ***iterative procedure under various assumptions of*** $\mathbf{l}_\mathbf{x}$ ***- curve.***

The iteration process is hinged on the basic mathematical identity (9) that exists in any life table where $_5\mathbf{s}_{\mathbf{x+}} = \mathbf{T}_{\mathbf{x+5}} / \mathbf{T}_\mathbf{x}$ and $_5\mathbf{E}_\mathbf{x} = \mathbf{e}^\mathbf{0}_{\mathbf{x+5}} / \mathbf{e}^\mathbf{0}_\mathbf{x}$. The iterative procedure makes use of the identity (9.1) with a known set of $_5\mathbf{s}_{\mathbf{x+}}$ values together with an initial approximation to the set of $_5\mathbf{E}_\mathbf{x}$ values over ages. Henceforth, the $_5\mathbf{E}_\mathbf{x}$ values will be simply called **'E-values'**. Corresponding to an initial approximation to the set of **'E-values'** over ages, somehow obtained, the first approximated set of **'p-values'** can be obtained through the equation (9.1) for a known set of $_5\mathbf{s}_{\mathbf{x+}}$ values over ages. A revised set of **'E-values'** can be computed from the **e(x)** column of the first approximated life table constructed on the basis of the first approximated set of **'p-values'** over ages. This revised set of **'E-values'** together with $_5\mathbf{s}_{\mathbf{x+}}$ values when applied to the basic identity (9.1) produces the second approximated set of **'p-values'** which, in turn, generates the second approximated life table -- **LT(2).** This **LT(2)** will yield the second revised set of **'E-values'** which can be used to obtain another new set of **'p-values'** through the equation (9.1) and, hence, the corresponding approximated life table. This procedure may be repeated in the following cyclical order, **-- 'p-values', LT, 'E-values'** -- as many time as required until the two consecutive approximations to the set of **'E-values'** at each and every age become sufficiently close to each other. The last entity of the initial of set of $_5\mathbf{E}_\mathbf{x}$ values sets tune in determining the survival probabilities at Adult ages uniquely. A method of estimation of the last entity of the initial set $\{_5\mathbf{E}^{(\mathbf{0})}_\mathbf{x}\}$ has also been proposed

An interesting feature of the iterative procedure is that the last entity, that is ${}_5\mathbf{E}^{(0)}_{w-5}$, of the initial **'E-column'** remains unaltered over iterations, whereas the other entities of the **'E-column'** continue to get modified until they converge or (practically) stabilize to fixed values sequentially one after another from the highest to the lowest age. The convergence is much faster when the iteration process is carried out under the Greville's approximation. It is worth mentioning here that whatever be the entities of the initial **'E-column'** the set of ${}_5\mathbf{s}_{x+}$ values computed from the final iterated life table becomes exactly identical to that of the **'true'** ${}_5\mathbf{s}_{x+}$ values over ages; however, the set of **'p-values'** corresponding to the final iterated life table will not be identical to the true ***'p-column' (p-column of the known life table)*** unless the last entity of the initial **'E-column'** is exactly equal to that of the known life table. An analytical proof of convergence of the process under the conventional approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$ can be shown easily (not shown here, but will be supplied on request). The analytical proof of convergence may also be extended under the Greville's approximation for ${}_5\mathbf{L}_x$ from $\mathbf{l}_x$. However, the formulation of the mathematical treatment becomes quite complicated**.**

Some Specific Observations and Direction for Future Research:

An empirical investigation based on Coale-Demeny regional model life tables shows that the value of ${}_5\mathbf{E}_x$, which can also be expressed as the ratio ${}_5\mathbf{s}_{x+}/{}_5\mathbf{p}_x$, remains almost unaltered over a broad range of mortality levels ($\mathbf{e}^0_0$) within a particular model mortality pattern even though the individual components of the ratio, that is ${}_5\mathbf{s}_{x+}$ and ${}_5\mathbf{p}_x$, vary significantly over the mortality levels (Lahiri, 1983). Similar feature of such an invariant property of the ${}_5\mathbf{E}_x$ values is also found with respect to UN new model life table system for developing countries (United Nations, 1982). Another interesting feature of the ${}_5\mathbf{E}_x$-column is that though its magnitude over ages and age-pattern remains almost steady over various mortality levels within a particular model mortality pattern but they vary considerably over the various model mortality patterns. This invariant property of the ${}_5\mathbf{E}_x$-column over various mortality levels within a model mortality pattern as mentioned above may be used to identify the mortality pattern of an individual life table. The basic concept of the iterative procedure, which use of the fundamental identity ${}_5\mathbf{p}_x = {}_5\mathbf{s}_{x+}/{}_5\mathbf{E}_x$ for given set of ${}_5\mathbf{s}_{x+}$ values, may be used for developing a model life table system for countries with limited and defective data.

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APPENDIX-A

Analytical Justification for the Algebraic Chain Relationship Between Two Consecutive ${}_5\mathbf{p}_x$'s as Shown Under the Formula (2) or (3) in the text.

Using the standard relationship between ${}_5\mathbf{L}_x$ and $\mathbf{T}_x$ columns of an abridged life table, that is, ${}_5\mathbf{L}_x = \mathbf{T}_x - \mathbf{T}_{x+5}$, it can be easily shown that **5-LSR,** that is, ${}_5\mathbf{s}_{x+2.5}\ (= {}_5\mathbf{L}_{x+5} / {}_5\mathbf{L}_x)$ can be expressed in terms of **5-cum-LSR**, that is, ${}_5\mathbf{s}_{x+}\ (= \mathbf{T}_{x+5} / \mathbf{T}_x)$ in the two successive age-groups through the following ***exact relationship***:

$$ {}_5\mathbf{s}_{x+2.5} = \frac{{}_5\mathbf{L}_{x+5}}{{}_5\mathbf{L}_x} = \frac{{}_5\mathbf{s}_{x+} * \left(1 - {}_5\mathbf{s}_{(x+5)+}\right)}{1 - {}_5\mathbf{s}_{x+}} \text{-------------(A.1)} $$

for x = 5, 10, 15, ………..,w-10, w-5.

The above identity shows that knowing the true values of ${}_5\mathbf{s}_{x+}$, how one can obtain the true values of ${}_5\mathbf{s}_{x+2.5}$ which determine a life table uniquely provided, of course, an initial value of ${}_5\mathbf{L}_x$ (say at age x = a) is known in advance. Alternatively, the true values of ${}_5\mathbf{s}_{x+2.5}$ can be used directly to estimate ${}_5\mathbf{p}_x$ under certain assumptions of $\mathbf{l}_x$-curve as discussed below:

Under the assumption of linearity of $\mathbf{l}_x$ in the two consecutive age-intervals (x, x+5) and (x+5, x+10), and ${}_5\mathbf{L}_x$ can be conventionally expressed as:

$$ {}_5\mathbf{L}_y \approx \tfrac{5}{2}(\mathbf{l}_y + \mathbf{l}_{y+5}), \textbf{ for } \mathbf{y = x \ \& \ x+5} \ \textbf{…………………….(A.2)} $$

Now, keeping in mind ${}_5\mathbf{p}_{x+5} = \mathbf{l}_{x+10} / \mathbf{l}_{x+5}$ by definition, one can easily find $1 + {}_5\mathbf{p}_{x+5}$ can be expressed as follows under the approximation (A.2):

$$ 1 + {}_5\mathbf{p}_{x+5} \approx {}_5\mathbf{s}_{x+2.5} * \frac{1 + {}_5\mathbf{p}_{x+5}}{{}_5\mathbf{p}_x}, \textbf{ for } \mathbf{x \geq 5} \ \textbf{…………(A.3)} $$

Now using (A.1) in (A.3) and rearranging the terms and simplifying the result we get the algebraic relationships mentioned in the text under the formulas (2) and (3). Under the Greville's approximation for ${}_5\mathbf{L}_x$ in terms $\mathbf{l}_x$, which assumes $\mathbf{l}_x$ is a cubic curve; it can be shown that ${}_5\mathbf{p}_x$ can be expressed by the relationship of the form mentioned under the approximation (12) in the main text.

Table 1:

Percentage Error in Estimating ${}_5p_x$ Values from ${}_5s_{x+}$ Values Alone Through the Formula (6) Compared to the True Values of ${}_5p_x$ Based on Coale-Deminy West Model Life Tables for Females.

	Level-5				Level-13				Level-21			
Age	${}_5s_{x+}$ (a)	${}_5\hat{p}_x$ (b)	${}_5p_x^t$ (c)	${}_5ER_x$ (d)	${}_5s_{x+}$ (a)	${}_5\hat{p}_x$ (b)	${}_5p_x^t$ (c)	${}_5ER_x$ (d)	${}_5s_{x+}$ (a)	${}_5\hat{p}_x$ (b)	${}_5p_x^t$ (c)	${}_5ER_x$ (d)
5	0.887576	0.905025	0.949841	4.7183	0.911440	0.962639	0.978643	1.6353	0.926347	0.995257	0.996620	0.1368
10	0.879013	0.955058	0.960798	0.5974	0.904681	0.980714	0.983408	0.2739	0.920734	0.996706	0.997277	0.0572
15	0.868563	0.948926	0.948833	0.0098	0.896700	0.977395	0.977390	0.0004	0.914214	0.995586	0.995647	0.0062
20	0.857354	0.937671	0.936111	0.1666	0.887760	0.971820	0.971152	0.0687	0.906662	0.993898	0.993758	0.0141
25	0.844868	0.928369	0.928445	0.0081	0.877455	0.967258	0.967326	0.0071	0.897764	0.992387	0.992418	0.0031
30	0.830330	0.919990	0.919304	0.0745	0.865203	0.963012	0.962959	0.0055	0.887069	0.990686	0.990943	0.0259
35	0.812905	0.912242	0.911583	0.0723	0.850315	0.958469	0.958491	0.0022	0.874006	0.988013	0.988408	0.0400
40	0.790887	0.905320	0.905262	0.0064	0.831697	0.953100	0.953562	0.0484	0.857814	0.983514	0.984214	0.0711
45	0.761577	0.893376	0.897865	0.5000	0.807760	0.943914	0.946242	0.2461	0.837451	0.976195	0.977053	0.0878
50	0.723219	0.869391	0.868749	0.0739	0.776877	0.927570	0.928313	0.0800	0.811405	0.964990	0.966106	0.1238
55	0.673645	0.829448	0.834354	0.5881	0.736647	0.900924	0.904696	0.4170	0.777336	0.947099	0.949576	0.2609
60	0.611406	0.771086	0.763477	0.9966	0.684167	0.859862	0.859964	0.0119	0.731963	0.917439	0.921162	0.4042
65	0.534504	0.695269	0.691701	0.5158	0.615384	0.800043	0.802014	0.2458	0.671204	0.888304	0.872605	0.4930
70	0.438649	0.598746	0.576436	3.8703	0.524821	0.715506	0.708255	1.0238	0.590792	0.792052	0.792685	0.0799
75	0.324744	-	-	-	0.405498	-	-	-	0.487299	-	-	-

Note: (a) ${}_5s_{x+} = \frac{T_{x+5}}{T_x}$; (b) Estimated value of ${}_5p_x$ obtained through the formula (6); (c) The symbol ${}_5p_x^t$ stands for true value of ${}_5p_x$ borrowed from the concerned West Model Life Table; (d) ${}_5ER_x = \frac{\left|{}_5\hat{p}_x - {}_5p_x^t\right|}{{}_5\hat{p}_x} \times 100$

Table 2

Minimum, Maximum and Range of Variation of the Percentage Errors in Estimating the Values of $_5p_x$, x=5, 10…70 & 75 through the Formulas (2) & (3) Knowing an Initial Value of $_5p_a$ (a=15, 20….., 35 & 40) through the Use of Formula (6) Based on West Model Life Tables for Females over various mortality Levels.

Age	Level-5			Level-15			Level-21		
a[1]	Minimum	Maximum	Range[2]	Minimum	Maximum	Range[2]	Minimum	Maximum	Range[2]
15	0.00889	0.10622	0.09733(2)	0.00002	0.00331	0.00329(1)	0.00610	0.01310	0.00703(2)
20	0.14378	2.02435	1.88057(6)	0.06524	0.28785	0.22261(6)	0.01407	0.03130	0.01723(3)
25	0.00728	0.12081	0.11353(3)	0.00707	0.03275	0.02566(4)	0.00254	0.00500	0.00246(1)
30	0.05619	0.80547	0.74928(5)	0.00480	0.02348	0.01868(3)	0.02541	0.05981	0.03440(4)
35	0.04826	0.65944	0.61118(4)	0.00114	0.00893	0.00779(2)	0.03829	0.08706	0.04877(5)
40	0.00415	0.07993	0.07578(1)	0.03871	0.17401	0.13530(5)	0.06855	0.15949	0.09094(6)

Notes: [1] The age '**a**' refers to the beginning of the age-interval (a, a+5) for which the probability of survival ($_5p_a$) is required to estimate first through the formula (6) so as obtain other values of $_5p_x$**'s** through the repeated application of the formulas (2) and (3).

[2] The figure within bracket attached to the range denotes the rank assigned according to ascending order of the magnitude of the range for a particular mortality level

Table 3

Values of ${}_5p_x$ at ages 5 and above estimated through the simultaneous use of the formulas (2) and (3) after obtaining the values of ${}_5p_{15}$ through the formula (6), and the corresponding relative percentage errors (${}_5ER_x$) in estimating ${}_5p_x$ based on West Model Life Tables for females over some selected mortality levels.

Age	Level-5		Level-19		Level-21	
a	${}_5\hat{p}_x$	${}_5ER_x$	${}_5\hat{p}_x$	${}_5ER_x$	${}_5\hat{p}_x$	${}_5ER_x$
5	0.949926	0.00889	0.978643	0.00002	0.996538	0.00625
10	0.960706	0.00961	0.983404	0.00036	0.997340	0.00632
15	0.948926	0.00982	0.977395	0.00047	0.995586	0.00614
20	0.936006	0.01127	0.971143	0.00091	0.993819	0.00610
25	0.928557	0.01204	0.967341	0.00152	0.992351	0.00672
30	0.919180	0.01346	0.962949	0.00101	0.991008	0.00655
35	0.911712	0.01421	0.958497	0.00062	0.988342	0.00671
40	0.905142	0.01520	0.953558	0.00040	0.984289	0.00760
45	0.898004	0.01549	0.946234	0.00079	0.976980	0.00752
50	0.868614	0.01551	0.928320	0.00072	0.966171	0.00674
55	0.834500	0.01740	0.904706	0.00111	0.949508	0.00717
60	0.763309	0.02204	0.859944	0.00234	0.921234	0.00787
65	0.691909	0.03012	0.802032	0.00221	0.872536	0.00788
70	0.576150	0.04966	0.708232	0.00331	0.792752	0.00849
75	0.443487	0.10622	0.583719	0.00189	0.673927	0.01310

Note: ${}_5ER_x = \dfrac{\left|{}_5\hat{p}_x - {}_5p_x^t\right|}{{}_5\hat{p}_x} \times 100$, where ${}_5p_x^t$ and ${}_5\hat{p}_x$ stand for the true and estimated values of ${}_5p_x$'s respectively. The true values of ${}_5p_x$'s are shown in Table-1.

Table 4
Set of $_5p_x$ Values of Various Iterations (under conventional approximation for $_5L_x$ from l_x) where the True Set of $_5s_{x+}$ Values Corresponds to W.M.L.-14* and All the Entities of the Initial Set of $_5E_x$ Values are same of those of W.M.L.-14

Age	True values $_5p_x$, $_5s_{x+}$ & $_5E_x$ According to C-D W.M.L-14			Estimated Values of $_5p_x$, $_5s_{x+}$ & $_5E_x$ at Iteration No. 1			Estimated Values of $_5p_x$, $_5s_{x+}$ & $_5E_x$ at Final Iteration No. 3		
x	$_5p_x$	$_5s_{x+}$	$_5E_x$	$_5p_x$	$_5s_{x+}$	$_5E_x$	$_5p_x$	$_5s_{x+}$	$_5E_x$
5	0.98193	0.90996	0.92670	0.98193	0.90996	0.92670	0.98193	0.90996	0.92670
10	0.98681	0.90260	0.91466	0.98681	0.90260	0.91466	0.98681	0.90260	0.91466
15	0.98035	0.89386	0.91177	0.98035	0.89386	0.91177	0.98035	0.89386	0.91177
20	0.97209	0.88407	0.90946	0.97209	0.88407	0.90946	0.97209	0.88407	0.90946
25	0.96953	0.87269	0.90012	0.96953	0.87269	0.90012	0.96953	0.87269	0.90012
30	0.96517	0.85888	0.88988	0.96517	0.85888	0.88988	0.96517	0.85888	0.88988
35	0.95837	0.84197	0.87854	0.95837	0.84197	0.87854	0.95837	0.84197	0.87854
40	0.94798	0.82107	0.86613	0.94798	0.82107	0.86613	0.94798	0.82107	0.86613
45	0.93446	0.79485	0.85060	0.93446	0.79485	0.85060	0.93446	0.79485	0.85060
50	0.91194	0.76163	0.83517	0.91194	0.76163	0.83517	0.91194	0.76163	0.83517
55	0.88189	0.71907	0.81537	0.88189	0.71907	0.81537	0.88189	0.71907	0.81537
60	0.83313	0.66438	0.79745	0.83313	0.66438	0.79745	0.83313	0.66438	0.79745
65	0.76801	0.59409	0.77354	0.76801	0.59409	0.77354	0.76801	0.59409	0.77354

* Note: Based on Coale-Demeny West Model Life Table for males at Level-14

Table 5

Set of ${}_5p_x$ Values of various iterations (under Conventional Approximation for ${}_5L_x$ from l_x) where the True set of ${}_5s_{x+}$ Values corresponds to W.M.L. – 14 and all the entries of the initial set of ${}_5E_x$ Values are all different from those W.M.L-14.

Age x	True values of (According to W.M. L. –14[a]		The values of ${}_5p_x$ at various iterations					Final iterations[b] No. 461	
			Iteration number						
	${}_5p_x$	${}_5s_{x+}$	1	101	201	301	401	${}_5p_x$	${}_5s_{x+}$
5	0.98193	0.90995768	0.90996	0.97934	0.98441	0.97657	0.97145	0.97123	0.90996
10	0.98681	0.90259698	0.90260	0.98244	0.98131	0.99626	0.99784	0.99787	0.90260
15	0.98035	0.89385593	0.89386	0.97656	0.97966	0.96958	0.96930	0.96929	0.89386
20	0.97209	0.88407116	0.88407	0.97563	0.97965	0.98340	0.98344	0.98344	0.88407
35	0.96953	0.87269469	0.87269	0.97790	0.95879	0.95801	0.95801	0.95801	0.87269
30	0.96517	0.85888091	0.85888	0.95869	0.97708	0.97718	0.97718	0.97718	0.85888
35	0.95837	0.84196573	0.84197	0.95386	0.94612	0.94612	0.94612	0.94612	0.84197
40	0.94798	0.82107158	0.82107	0.95918	0.96086	0.96086	0.96086	0.96086	0.82106
45	0.93446	0.79484948	0.79485	0.92132	0.92115	0.92115	0.92115	0.92115	0.79485
50	0.91194	0.76162516	0.76163	0.92620	0.92621	0.92621	0.92621	0.92621	0.76162
55	0.88189	0.71906699	0.71907	0.86671	0.86671	0.86671	0.86671	0.86671	0.71907
60	0.83313	0.66438071	0.66438	0.85019	0.85019	0.85019	0.85019	0.85019	0.66438
65	0.76801	0.59408692	0.74865[c]	0.74865[c]	0.74865[c]	0.74865[c]	0.74865[c]	0.74865[c]	0.59409

Note: (a) For details of the life-table W.M.L. –14, see the text.

(b) Though the process conforms to the true set of ${}_5s_{x+}$ values, the final iterated set of ${}_5p_x$ values is entirely different from that of the W.M. L-14.

(c) The initial value of ${}_5s_{65+}$, which is different from that of the true set, remains unaltered over iterations. The other entities of the set of ${}_5p_x$'s get stabilized one after another from higher to lower ages at different stages of iteration.

Table 6

Set of $_5p_x$ - values at various iterations (under Conventional Approximation for $_5L_x$ from l_x) where the True set of $_5s_{x+}$ Values corresponds to W.M.L. – 14 and the last entity of the initial set of $_5E_x$ Values is exactly identical to W.M.L.-14 whereas the other entities of the initial $_5E_x$ is are all equal to unity.

Age x	True values of (According to W.M. L. –14[a])		The values of $_5p_x$ at various iterations					Final iterations[b] No.450	
			Iteration number						
	$_5p_x$	$_5s_{x+}$	1	101	201	301	401	$_5p_x$	$_5s_{x+}$
5	0.98193	0.90995768	0.90996	0.97866	0.98225	0.98492	0.98209	0.98196	0.90996
10	0.98681	0.90259698	0.90260	0.98229	0.98014	0.98581	0.98678	0.98680	0.90260
15	0.98035	0.89385593	0.89386	0.97761	0.98630	0.98055	0.98035	0.98035	0.89386
20	0.97209	0.88407116	0.88407	0.97738	0.96963	0.97205	0.97208	0.97208	0.88407
35	0.96953	0.87269469	0.87269	0.97697	0.97011	0.96955	0.96954	0.96954	0.87269
30	0.96517	0.85888091	0.85888	0.95362	0.96509	0.96517	0.96517	0.96517	0.85888
35	0.95837	0.84196573	0.84197	0.96414	0.95838	0.95837	0.95837	0.95837	0.84197
40	0.94798	0.82107158	0.82107	0.94667	0.94798	0.94798	0.94798	0.94798	0.82107
45	0.93466	0.79484948	0.79485	0.93461	0.93446	0.93446	0.93446	0.93446	0.79485
50	0.91194	0.76162516	0.76163	0.91192	0.91192	0.91192	0.91192	0.91192	0.76163
55	0.88189	0.71906699	0.71907	0.88190	0.88189	0.88189	0.88189	0.88189	0.71907
60	0.83313	0.66438071	0.66438	0.83312	0.83312	0.83312	0.83312	0.83312	0.66438
65	0.76801	0.59408692	0.76801[c]	0.76801[c]	0.76801[c]	0.76801[c]	0.76801[c]	0.76801[c]	0.59409

Note: (a) For details of the life-table W.M.L. –14 see the text.

(b) The process conforms to the true set of $_5p_x$ and $_5s_{x+}$ values as in W.M. L-14.

(c) The initial value of $_5p_{65}$, which is only value of $_5p_x$, is identical to that of the true set of $_5p_x$'s, remains unaltered over the iterations. The other entities of the set of $_5p_x$'s get stabilized one after another from higher to lower ages at different stages of iterations.

Table 7

Set of Observed (Estimated) ${}_5p_x$ Values at Various Iterations (under Greville's approximation for ${}_5L_x$ from l_x) where the true set of ${}_5s_{x+}$ values corresponds to W.M.L -12 (GREV)* and All the Entities of the Initial Set of ${}_5E_x$ Values are same of those of W.M.L -12 (GREV)

Age	True values ${}_5p_x$, ${}_5s_{x+}$ & ${}_5E_x$ [According to W.M.L-12 (GREV)]			Estimated Values of ${}_5p_x$, ${}_5s_{x+}$ & ${}_5E_x$ Values at the First & Final Iteration		
x	${}_5p_x$	${}_5s_{x+}$	${}_5E_x$	${}_5p_x$	${}_5s_{x+}$	${}_5E_x$
4	0.97031	0.90681	0.93455	0.97031	0.90681	0.93455
9	0.98302	0.89946	0.91501	0.98302	0.89946	0.91500
14	0.97769	0.89031	0.91063	0.97769	0.89031	0.91063
19	0.96737	0.88018	0.90987	0.96737	0.88018	0.90987
24	0.96168	0.86874	0.90335	0.96168	0.86874	0.90336
29	0.95869	0.85490	0.89174	0.95869	0.8549	0.89174
34	0.95039	0.83793	0.88167	0.95039	0.83793	0.88167
39	0.93906	0.81723	0.87026	0.93906	0.81723	0.87026
44	0.92581	0.79134	0.85476	0.92582	0.79134	0.85475
49	0.90306	0.75868	0.84012	0.90305	0.75868	0.84013
54	0.87439	0.71694	0.81993	0.87440	0.71694	0.81992
59	0.82685	0.66339	0.80232	0.82685	0.66339	0.80231
64	0.76346	0.59493	0.77926	0.76345	0.59493	0.77927

* Note: Based on Life Table constructed under Greville's approximation for ${}_5L_x$ after estimating ${}_5p_x$ - values at ages 4, 9, 14,……., 64 and 69 corresponding to Coale-Demeny West Model Life Table for males at Level 12 (W.M.L.-12)

Table 8:
Set of ${}_5p_x$ - values of various iterations (under Greville's Approximation for ${}_5L_x$ from l_x) where the True set of ${}_5s_{x+}$ Values corresponds to W.M.L. – 12 (GREW) and all entities, excepting the last entity which is sufficiently close to the true one of the initial set of ${}_5E_x$ Values are all unity.

Age x	True values of (According to W.M. L. –12 (GREW)[a]		The values of ${}_5p_x$ at various iterations				Final iterations[b] No.328	
			Iteration number					
	${}_5p_x$	${}_5s_{x+}$	1	101	201	301	${}_5p_x$	${}_5s_{x+}$
4	0.97031	0.90680595	0.90681	0.97396	0.97860	0.97091	0.97082	0.90680678
9	0.98302	0.89946477	0.89946	0.98340	0.98046	0.98270	0.98267	0.89946514
14	0.97769	0.89031362	0.89031	0.97299	0.97823	0.97806	0.97806	0.89031368
19	0.96737	0.88017914	0.88018	0.96088	0.96706	0.96699	0.96699	0.88017915
24	0.96168	0.86873693	0.86874	0.97161	0.96206	0.96206	0.96206	0.86873693
29	0.95869	0.85490471	0.85490	0.95514	0.95829	0.95829	0.95829	0.85490471
34	0.95039	0.83793031	0.83793	0.95080	0.95080	0.95080	0.95080	0.83793031
39	0.93906	0.81722532	0.81723	0.93871	0.93863	0.93863	0.93863	0.81722532
44	0.92582	0.79134284	0.79134	0.92628	0.92628	0.92628	0.92628	0.79134284
49	0.90305	0.75867836	0.75868	0.90257	0.90257	0.90257	0.90257	0.75867836
54	0.87440	0.71693938	0.71694	0.87493	0.87493	0.87493	0.87493	0.71693938
59	0.82685	0.66339171	0.66339	0.82627	0.82627	0.82627	0.82627	0.66339171
64	0.76345	0.59493299	0.76441[c]	0.76441[c]	0.76441[c]	0.76441[c]	0.76441[c]	0.59493299

Note: (a) For details of the life-table W.M.L.–12, GREW), see the text.

(b) Though the process conforms exactly to the true set of ${}_5s_{x+}$ values, the final iterated set of ${}_5p_x$ values though not identical but becomes quite close to that of W.M. L-12 (GREW) as the last entity of the initial set of ${}_5E_x$ values is close enough to that of W.M. L-12 (GREW).

(c) The initial value of ${}_5p_{64}$ which is quite close to that of true set, remains unaltered over the iterations. The other entities of the set of ${}_5p_x$'s also get stabilized at some closer values top those of the true set one after another from higher to lower ages at different stages of iterations.

Table 9
Set of ${}_5p_x$ - values of various iterations (under Greville's Approximation for ${}_5L_x$ from l_x) where the True set of ${}_5s_{x+}$ Values corresponds to W.M.L. 12 (GREV) and the last entity of the initial set of ${}_5E_x$ Values are exactly identical to that of W.M.L. 12 (GREV) whereas the other entities of the initial ${}_5E_x$ are all equal to unity

Age x	True values of (According to W.M. L. –12 (GREW)[a]		The values of ${}_5p_x$ at various iterations				Final iterations[b] No.328	
			Iteration number					
	${}_5p_x$	${}_5s_{x+}$	1	101	201	301	${}_5p_x$	${}_5s_{x+}$
4	0.97031	0.90680595	0.90681	0.97396	0.97840	0.97041	0.97032	0.90680678
9	0.98302	0.89946477	0.89946	0.98343	0.98079	0.98305	0.98302	0.89946514
14	0.97769	0.89031362	0.89031	0.97304	0.97786	0.97770	0.97769	0.89031369
19	0.96737	0.88017914	0.88018	0.96087	0.96743	0.96737	0.96737	0.88017915
24	0.96168	0.86873693	0.86874	0.97139	0.96167	0.96168	0.96168	0.86873693
29	0.95869	0.85490471	0.85490	0.95551	0.95869	0.95869	0.95869	0.85490471
34	0.95039	0.83793031	0.83793	0.95038	0.95039	0.95039	0.95039	0.83793031
39	0.93906	0.81722532	0.81723	0.93914	0.93906	0.93906	0.93906	0.81722532
44	0.92582	0.79134284	0.79134	0.92583	0.92582	0.92582	0.92582	0.79134284
49	0.90305	0.75867836	0.75868	0.90305	0.90305	0.90305	0.90305	0.75867836
54	0.87440	0.71693938	0.71694	0.87440	0.87440	0.87440	0.87440	0.71693938
59	0.82685	0.66339171	0.66339	0.82685	0.82685	0.82685	0.82685	0.66339171
64	0.76345	0.59493299	0.76345[c]	0.76345[c]	0.76345[c]	0.76345[c]	0.76345[c]	0.59493299

Note: (a) For details of the life-table W.M.L.–12, GREW) in the text.

(b) The process conforms to the true set of ${}_5p_x$ of ${}_5s_{x+}$ as in W.M. L-12 (GREW).

(c) The initial value of ${}_5p_{64}$, which is the only value of ${}_5p_x$, is identical to that of the true set of ${}_5p_x$, remains unaltered over the iterations. The other entities of the set of ${}_5p_x$'s get stabilized one after another from higher to lower ages at different stages of iterations.

FLOW CHARAT OF THE ITERATION PROCESS

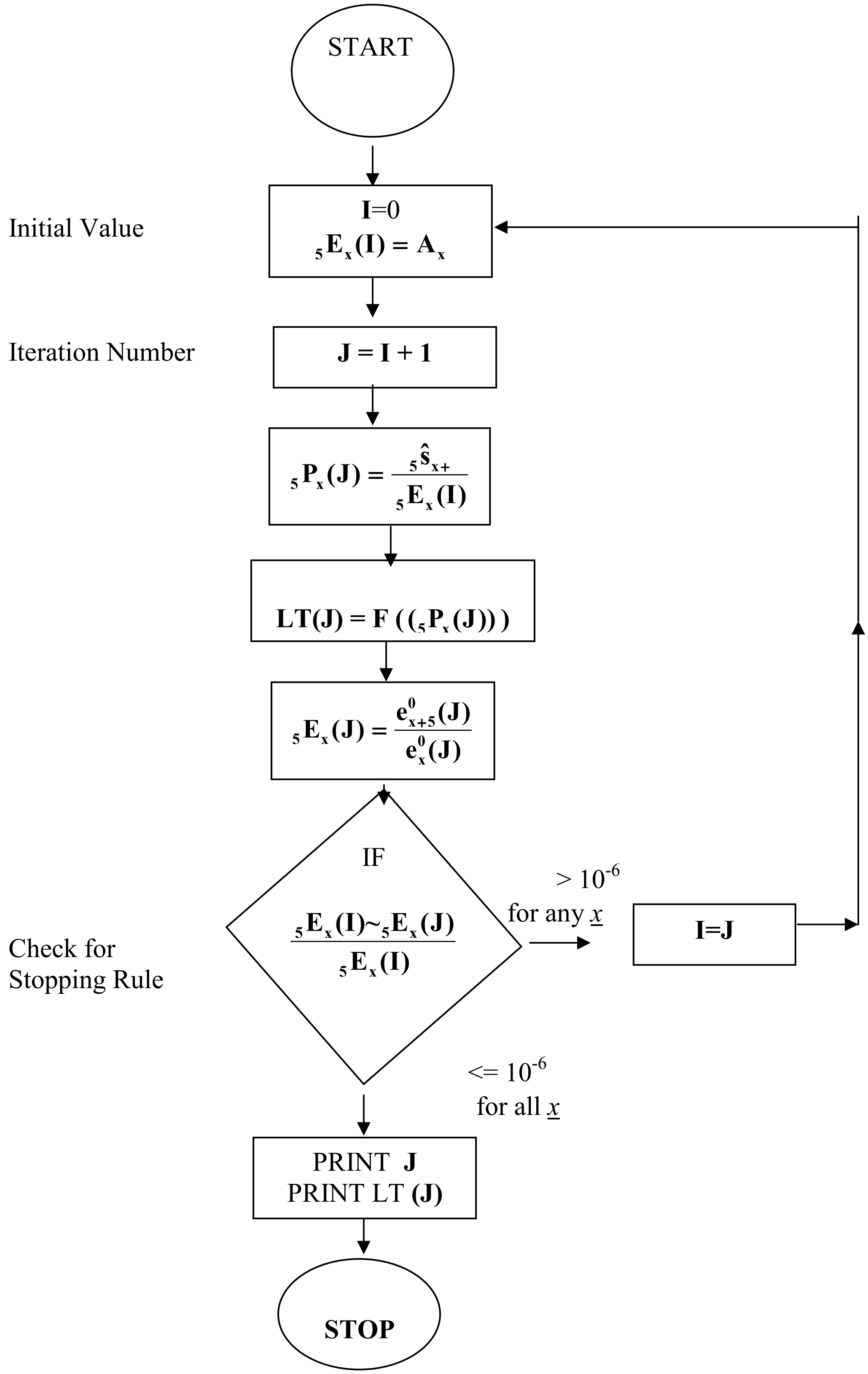

DIAGRAM-1